COMPLETE SCHOOL ALGEBRA

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COMPLETE SCHOOL ALGEBRA

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PREFACE

This book, designed for at least one and one-half year's work in algebra, is intended for the use of those schools which find a one-book course best adapted to their needs. Experience thoroughly justifies a careful review at the beginning of the third half year's work. As the first twenty-three chapters contain the greater portion of the usual first year's work in algebra, it seemed best to follow them with the review material. Here each topic is given a broader and more advanced treatment than is permissible in first-year work. New matter is used throughout, and many new applications are given in order to make a fresh and inviting appeal to the student. In the remaining chapters the aim has been to include those topics considered necessary by the best secondary schools, and to treat each in a clear, practical, and attractive manner.

A painstaking attempt has been made in this book to select and arrange the material so that no student will find the work insipid, while the student who might find the subject unduly hard will be prepared for approaching difficulties, and encouraged into interested and successful effort. To accomplish this, every group of problems or exercises is graded with exactness; lists of exercises which afford extended practice in the mere translation of English into algebraic language are given at proper intervals; the explanations are many and unusually full; and the exceptionally numerous hints and illustrative examples are placed precisely where needed.

The authors do not believe it is desirable to attempt to correlate algebra with other subjects than arithmetic, geometry, and physics. They have no hobbies in the way of "practical" problems, no aversion to those that are informational. They have, however, a strong preference for a problem that is "thinkable." Experience shows that long lists of problems manufactured from tables of statistics are not necessarily interesting. It will be found that there is a close connection with geometry through the problems on factoring, proportion, variation, quadratic equations, and the exercises and problems on radicals. A correlation with physics equally close is impossible and undesirable. Yet at many places there is definite preparation for the point of view needed by the student of physics.

The exercises and problems have been tested by class use. While their number is ample it is not excessive. On account of the grading, omissions should preferably be made from the ends of the various lists.

Particular attention is called to certain other important features, namely: the prominence everywhere given to the equation; the continued emphasis on checking; the numerous problems and their varied character; the frequent changes from exercises to problems; the short reviews in factoring; the extent of the work with graphs, — its detailed explanation, its position before the process it illustrates, its use in connection with logarithms; the introduction of the idea of a function and its graphical treatment; the accuracy of the definitions; the historical notes; the portraits of eminent mathematicians; and the biographical notes which accompany them.

Of late years the teaching of algebra has been widely and thoughtfully discussed. Every development of this period has been carefully studied by the authors. It has been their object to discern all the sound, progressive ideas in the recent evolution and to utilize them in the preparation of a text which will develop both the student's reasoning power and the needful facility on the technical side of algebra. Other constant aims have been to make the book teachable, and to reduce the explanation on the teacher's part to a minimum. Scrupulous care was bestowed on the English, in order to give the definitions accurately, to state the problems clearly, and to formulate the rules with simplicity and precision. Along with the endeavor to accomplish these various ends, a continuous effort has been made to produce a text which is distinctly modern, lucid, interesting, and mathematically correct.

The present volume is in many parts identical with either our "First Course in Algebra" or our "Second Course in Algebra," in which a detailed acknowledgment is made of our obligation to those who have given us the benefit of their criticism and suggestion. That acknowledgment will not be repeated here, but to all of the many teachers who have so materially aided our efforts, we wish to express our hearty thanks.

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COMPLETE SCHOOL ALGEBRA

CHAPTER I.

INTRODUCTION TO ALGEBRA

1. **Algebra** deals with many topics which are new to the student, yet he will find the subject a continuation of his previous work in arithmetic.

2. **Symbols.** Symbols are employed far more extensively in algebra than in arithmetic, and many new ideas arise in connection with their meaning and use. Some symbols represent numbers, others indicate operations upon them, and others represent relations between numbers. Arabic numerals, and letters are used to represent numbers. The following symbols of operation, $+$, $-$, $\times$, and $\div$, have the same meaning as in arithmetic. The sign $+$ is read *plus;* $-$, *minus;* $\times$, *multiplied by;* and $\div$, *divided by.* The sign of multiplication is usually replaced by a dot or omitted. For example, $3 \times a$ is written $3 \cdot a$, or $3a$, and $2 \times a \times b$ is written $2ab$. Also $a \div b$ is often written $\frac{a}{b}$. The sign $=$ is read *equals,* or *is equal to.*

EXERCISES

1. Express $4h + 3m$ in seconds, if h and m stand for the number of seconds in an hour and in a minute respectively.

2. Express $5y + 4f$ in inches, if y and f stand for the number of inches in a yard and in a foot respectively.

3. If q and d represent the number of cents in a quarter and in a dime respectively, find the value in cents of $4q + 6d$.

4. If t and h represent the number of pounds in one ton and in one hundredweight respectively, express $4t + 6h$ in pounds.

5. $3x + 5x =$ how many x?

6. $4x + 5x = (?)\,x$.

7. $2x + 3x + 6x = ?$

8. $2x + 2 + 3x + 4 = (?)\,x + ?$

9. $x + x + 2 + x + x + 2 = ?$

10. $n + n + 1 + n + 2 = ?$

11. $5a + 18 - 3a - 7 = ?$

12. $8x - 3 + 18 - 5x = ?$

13. $4w - 8 + 3w + 20 = ?$

14. If m represents one month and $y = 12m$, express $y + 5m$ in terms of m.

15. If y (one yard) equals $3f$ ($f =$ one foot), express $2y + 7f$ in terms of f.

16. Express $4q + 3n$ in terms of n, if q (one quarter) equals $5n$ ($n =$ one nickel).

17. Express $2d + 15h$ in terms of h, if $d = 24h$.

18. Express $15h + 50m$ in terms of m, if $h = 60m$.

19. The side of a square is 5 inches. What is its area? its perimeter?

20. The side of a square is s inches. What represents its perimeter? its area?

21. The base of a rectangle is 12 feet and its altitude is 4 feet. What is its area? its perimeter?

22. If b represents the number of feet in the base of a rectangle and a the number of feet in its altitude, what is its area? its perimeter?

23. A rectangle is twice as long as it is wide. Let x represent the number of inches in its width. Then express (a) the length in terms of x; (b) the perimeter; (c) the area.

24. A man is three times as old as his son. If y is the number of years in the son's age, what will represent the father's age?

25. A father is 30 years older than his son. If y represents the son's age in years, what will represent the father's age?

26. A rectangle is 12 feet longer than it is wide. Let w represent the width in feet. Then represent the length and the perimeter in terms of w and some numbers.

27. A rectangle is 18 feet narrower than it is long. If w represents the width in feet, what will conveniently represent the length? the perimeter?

28. A rectangle is 4 feet longer than twice its width. Express the width, the length, and the perimeter in terms of a letter, or a letter and numbers.

Origin of symbols. Many of the symbols that are in common use in algebra at the present time have histories which not only are interesting in themselves, but which also serve to indicate the slow and uncertain development of the subject. It is often found that symbols which seem without meaning represent some abbreviation or suggestion long since forgotten, and that operations and methods which we find hard to master have sometimes required hundreds of years to perfect.

In the early centuries there were practically no algebraic symbols in common use; one wrote out in full the words *plus*, *minus*, *equals*, and the like. But in the sixteenth century several Italian mathematicians used the initial letters $\bar{p}$ and $\bar{m}$ for $+$ and $-$. Some think that our modern symbol $-$ came into use through writing the initial m so rapidly that the curves of the letter gradually flattened out, leaving finally a straight line. The symbol $+$ may have originated similarly in the rapid writing of the letter p. But in the opinion of others these symbols were first used in the German warehouses of the fifteenth century to mark the weights of boxes of goods. If a lot of boxes, each supposed to weigh 100 pounds, came to the warehouse, the weight would be checked, and if a certain box exceeded the standard weight by 5 pounds, it was marked $100+5$; if it lacked 5 pounds, it was marked $100-5$. Though the first book to use these symbols was published in 1489, it was not until about 1630 that they could be said to be in common use.

Both of the symbols for multiplication given in the text were first used about 1630. The cross was used by two Englishmen, Oughtred and Harriot, and the dot is first found in the writings of the Frenchman, Descartes. It is interesting to note that Harriot was sent to America in 1585 by Sir Walter Raleigh and returned to England with a report of observations. He made the first survey of Virginia and North Carolina and constructed maps of those regions.

It is strange that the line was used to denote division long before any of the other symbols here mentioned were in use. This is, in fact, one of the oldest signs of operation that we have. The Arabs, as early as 1000 A.D., used both $\frac{a}{b}$ and a/b to denote the quotient of a by b. The symbol $\div$ did not occur until about 1630.

Equality has been denoted in a variety of ways. The word *equals* was usually written out in full until about the year 1600, though the two sides of an equation were written one over the other by the Hindus as early as the twelfth century. The modern sign $=$ was probably introduced by the Englishman, Recorde, in 1557, because, he says, "Noe. 2. thynges can be moare equalle" than two parallel lines. This symbol was not generally accepted at first, and in its place the symbols $\|$, ∞, and $\propto$ are frequently met during the next fifty years.

3. The usefulness of symbols. Symbols enable one to abbreviate ordinary language in the solution of problems.

For example: Three times a certain number is equal to 20 diminished by 5. What is the number?

If n represents the number, the preceding statement and question can be written in symbols, thus:

$$3\,n = 20 - 5.$$

$$n = ?$$

The symbolic statement, $3\,n = 20 - 5$, is called an *equation* and n the *unknown number*.

$$\begin{aligned} &\text{If} & 3\,n &= 20 - 5, \\ && 3\,n &= 15, \\ &\text{and} & n &= 5. \end{aligned}$$

The preceding example illustrates the algebraic method of stating and solving the problem. The method is brief and direct and its advantages will become more apparent as the student progresses.

EXAMPLE

1. The sum of two numbers is 112. The greater is three times the less. What are the numbers?

Solution: By the conditions of the problem,

$$\text{greater number} + \text{less number} = 112.$$

If we represent the less by l, then $3l$ must represent the greater, and the above statement becomes

$$3l + l = 112.$$

Collecting, $4l = 112.$

Whence $l = \frac{112}{4} = 28,$

and $3l = 3 \times 28 = 84.$

Therefore the greater number is 84 and the less 28.

PROBLEMS

1. The sum of two numbers is 160. The greater is four times the less. Find each.

2. A certain number plus five times itself equals 216. Find the number.

3. One number is seven times another. Their sum is 72. Find each.

4. The first of three numbers is twice the third, and the second is four times the third. The sum of the three numbers is 105. Find the numbers.

5. The sum of three numbers is 117. The second is twice the first, and the third three times the second. Find each.

6. There are three numbers whose sum is 192. The first is twice the second, and the third equals the sum of the other two. Find the numbers.

7. The sum of three numbers is 324. The second is five times the first, and the third is six times the second. Find the numbers.

8. The sum of three numbers is 104. The second is three times the first, and the third is the sum of the other two. Find the numbers.

9. What sum of money placed at interest for 1 year at 5% amounts to \$378?

Solution: From arithmetic,

$$\text{principal} + \text{interest} = \text{amount}.$$

By the conditions of the problem,

$$\text{principal} + .05 \text{ principal} = \$378.$$

If p represents the principal, this last statement becomes

$$p + .05p = 378.$$

Collecting, $$1.05\,p = 378.$$

Whence $$p = \frac{378}{1.05} = 360.$$

Therefore the required sum is \$360.

10. What sum of money placed at interest for 1 year at 6% amounts to \$265?

11. What sum of money placed at simple interest for 3 years at 4% will amount to \$700?

12. In how many years will \$225, at 6% simple interest, gain \$27?

13. In how many years will \$520, at $6\frac{1}{2}$% simple interest, gain \$169?

14. At what per cent simple interest will \$375 gain \$75 in 2 years?

Solution: Let $x =$ the rate of interest.

Then $\$375\,x =$ the interest for one year,

and $\$750\,x =$ the interest for two years.

But $\$75 =$ the interest for two years.

Therefore $750\,x = 75,$

and $x = \frac{1}{10}.$

Hence the money is lent at 10%.

15. At what per cent simple interest will \$825 gain \$165 in 4 years?

16. At what per cent simple interest will \$250 amount to \$317.50 in 6 years?

17. In how many years will \$200 double itself at 4% simple interest?

18. In how many years will $150 treble itself at 5% simple interest?

19. The perimeter of a certain square is 160 feet. Find the length of each side.

20. The perimeter of a certain rectangle is 256 feet. It is three times as long as it is wide. Find its dimensions.

21. The perimeter of the rectangle formed by placing two equal squares side by side is 198 inches. Find the dimensions and the perimeter of each square.

22. Two equal squares are placed side by side, forming a rectangle. If the perimeter of each square is 120 inches, find the perimeter of the rectangle.

4. Representation of numbers. In algebra numbers are represented by one or more numerals or letters or by both combined.

Thus 3, 25, a, $2b$, $4xy$, and $2x + 3$ are algebraic symbols for numbers.

Precisely what numbers $4xy$ and $2x + 3$ represent is not known until the numbers which x and y stand for are known. In one problem these symbols may have values quite different from those they have in another. To devise methods of determining these values in the various problems which arise is the principal aim of algebra.

5. Factors. A **factor** of a product is any one of the numbers which multiplied together form the product.

Thus $3ab$ means 3 times a times b. Here 3, a, and b are each factors of $3ab$. Similarly the expression $4(a + b)$ means 4 times the sum of a and b. Here 4 and $a + b$ are factors of $4(a + b)$.

6. Exponents. An **exponent** is a number written at the right of and above another number to show how many times the latter is to be taken as a factor.

(Later this definition will be modified so as to include fractions and other algebraic numbers as exponents.)

Thus $3^2 = 3 \cdot 3$; $5^3 = 5 \cdot 5 \cdot 5$. Also $a^4 = a \cdot a \cdot a \cdot a$, and $4xy^3 = 4 \cdot x \cdot y \cdot y \cdot y$. In a^b, b is the exponent of a. If a is 4 and b is 3, $a^b = 4^3 = 4 \cdot 4 \cdot 4$. The exponent 1 is not usually written.

7. Coefficients. If a number is the product of two factors, either of these factors is called the **coefficient** of the other in that product.

Thus in $4x^2y$, 4 is the coefficient of x^2y, y is the coefficient of $4x^2$, and $4y$ is the coefficient of x^2. The numerical coefficient 1 is usually omitted. If a numerical coefficient other than 1 occurs, it is usually written first. For instance, we write $5x$, not $x5$.

The following examples illustrate the difference in meaning between a coefficient and an exponent:

$$3x = x + x + x.$$
$$x^3 = x \cdot x \cdot x.$$

If, in each case, $x = 5$, $3x$ stands for the number 15, while x^3 stands for 125. If x is 10 in each case, $3x = 30$, while $x^3 = 1000$.

8. Use of parentheses. If two or more numbers connected by signs of operation are inclosed in a parenthesis, the entire expression is treated as a symbol for a single number.

Thus $3(6 + 4)$ means $3 \cdot 10$, or 30; $(17 - 2) \div (8 - 3)$ means $15 \div 5$, or 3; $(5 + 7)^2$ means 12^2, or 144; and $6(x + y)$ means six times the sum of x and y.

As in arithmetic, the symbol for square root is $\sqrt{}$, or $\sqrt[2]{}$, and the symbol for cube root is $\sqrt[3]{}$.

The name **radical sign** is applied to all symbols like the following: $\sqrt{}$, $\sqrt[2]{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, etc. The small figure in a radical sign, like the 3 in $\sqrt[3]{}$, is often called the **index**.

Note. There has been a considerable variety in the symbols for the roots of numbers. The symbol $\surd$ was introduced in 1544 by the German, Stifel, and is a corruption of the initial letter of the Latin word *radix*, which means "root." Before his time square root was denoted by the symbol ℞, used nowadays by physicians on prescriptions as an abbreviation for the word *recipe*. Thus $\sqrt[4]{5}$ would have been denoted by ℞^{4}5. Some early writers used a dot to indicate square root, and expressed $\sqrt{2}$ by $\cdot 2$. The Arabs denoted the root of a number by an arabic letter placed directly over the number.

EXERCISES

Write in symbols:

1. The sum of three times a and five times b.

2. Three times a subtracted from five times b.

3. The square of a subtracted from the square of b.

4. Two times a squared subtracted from three times a squared.

5. The quotient of a divided by b.

6. The product of four times a squared and b.

7. The sum of a and b divided by their difference.

8. The product of a and $2b - c$.

9. The product of a by the sum of b and c.

10. The result of subtracting $a - b$ from $7x$.

11. The product of the sum of a and b by their difference.

12. The sum of the square root of $5a$ and the cube root of $7b$.

13. The product of $x - y$ and the square root of $7x$.

14. The square of the sum of a and b.

15. The square of the difference of a and b.

16. The quotient of three times a times the square of b by four times c times the cube of a.

17. The sum of the quotients of a by $3x$ and $4y$ by c.

18. Read Exercises 1–14, page 11.

9. Order of fundamental arithmetical operations. If we read the expression $6 + 4 \cdot 9 - 12 \div 3$ from left to right, and perform each indicated operation as we come to its symbol, we obtain 10, 90, 78, and a final result of 26. If we perform the multiplication and division first, the expression becomes $6 + 36 - 4$, which equals 38. These results show that the value of the expression is determined largely by the order in which the operations are performed. When there is no statement to the contrary, it is understood that:

I. *A series of operations involving addition and subtraction alone shall be performed in the order in which they occur.*

Thus $8 + 12 - 10 + 6 = 20 - 10 + 6 = 10 + 6 = 16$.
It is incorrect to say $8 + 12 - 10 + 6 = 20 - 16 = 4$.

II. *A series of operations involving multiplication and division alone shall be performed in the order in which they occur.*

Thus $8 \cdot 12 \div 6 \cdot 4 = 96 \div 6 \cdot 4 = 16 \cdot 4 = 64$.
It is incorrect to say $8 \cdot 12 \div 6 \cdot 4 = 96 \div 24 = 4$.

III. *In a series of operations involving addition, subtraction, multiplication, and division the multiplications and divisions shall be performed in order before any addition or subtraction. Then the additions and subtractions shall be performed in accordance with I.*

Therefore $6 + 4 \cdot 9 - 12 \div 3 = 6 + 36 - 4 = 38$.
It is incorrect to say $6 + 4 \cdot 9 - 12 \div 3 = 10 \cdot 9 - 4 = 86$.

In a series of operations an expression inclosed in a parenthesis is regarded as a single number. Obviously, within any parenthesis I, II, and III apply.

Therefore $(3 + 2)6 = 5 \cdot 6 = 30$,
and $8 + (7 - 3)(9 - 6 \div 2) - 4 = 8 + 4 \cdot 6 - 4 = 8 + 24 - 4 = 28$.

EXERCISES

Simplify the following:

1. $20 - 5 + 6 - 10$.

2. $16 - (8 - 2)$.

3. $14 - (16 - 8) + (12 - 4)$.

4. $6 \div 3 - 2$.

5. $8 \cdot 6 \div 3 - 10$.

6. $18 \div (2 \cdot 3)$.

7. $(6 - 3)(17 - 2 \cdot 5)$.

8. $23 - 2 \cdot 6 - 4 \div 2 + 16$.

9. $18 \div (9 - 3)$.

10. $(10 - 3)(16 - 3 \cdot 2 + 8 \div 4)$.

11. $14 - 3 \cdot (16 - 2 \cdot 5) \div 6 + 8 \cdot 2$.

12. $(18 - 2) - 6 \div (4 + 2 \cdot 8 - 18 \div 9)$.

13. $(16 - 6)(18 - 8) \div 100 \cdot 5 - 5$.

14. $(5 + 3)(5 - 3) \div 5 - 3$.

10. Evaluation of algebraic expressions. Finding the numerical value of an expression for certain values of the letters therein is frequently necessary. This process will be used later to test the accuracy of the results of algebraic operations.

EXERCISES

In Exercises 1–23 let $a = 3$, $b = 1$, $c = 5$, $d = 7$, and $e = 2$. Substitute for each letter its numerical value and then simplify the results according to the rules of § 9:

1. $4a + 3d$.
2. $a^2 + 2c$.
3. $ab + cd$.
4. $c^2 - 5ab$.
5. $abcd - 4e^2$.
6. $\dfrac{d+c}{e}$.
7. $\dfrac{cde}{5e} + \dfrac{ace}{2a}$.
8. $\dfrac{10b}{e} - \dfrac{c}{b}$.
9. $\dfrac{1}{c} + \dfrac{1}{d} + \dfrac{1}{e}$.
10. $\dfrac{e^2}{ac} + \dfrac{4cd}{ae}$.
11. $\dfrac{cd + ae + ce}{a - b + c}$.
12. $\dfrac{a^2 + b^2 + c^2 + d^2}{a + b + c + d}$.
13. $\dfrac{a^3 - ce^2 + 3cd}{d - a + c}$.
14. $5b^3 + 4b^2 - 2b - 5$.
15. $3b^5 - 14b^4 + 11b^3 + 11b^2 + 13b - 20$.
16. a^e.
17. c^a.
18. $d^e + e^a$.
19. $b^c + c^a$.
20. $d^e - c^2 + b^3$.
21. $e^2 \cdot c^a$.
22. $\dfrac{d + e^c}{2c + a}$.
23. $\dfrac{a^a + 3b}{c^e - d - e - b}$.

Find the numerical value of the following expressions when $a = 4$, $b = 0$, $c = 5$, $d = 7$, and $e = 8$:

24. $\dfrac{4a + 3b + 2d}{c + e + 2}$.
25. $\dfrac{b}{a + c + d}$.
26. $abc + acd - be$.
27. $\dfrac{ab}{c} + \dfrac{bd}{a} + \dfrac{be}{cd}$.

28. $\dfrac{a^2 - b^2 + c^3}{3d - 2e + c}$.

29. $\sqrt{a} + \sqrt{2e}$.

30. $2\sqrt{a} + \sqrt[3]{e}$.

31. $b\sqrt{a} + cd + \sqrt[3]{e} \cdot \sqrt{a}$.

32. $c\sqrt[2]{2ae} + d\sqrt[3]{2ae}$.

33. $\sqrt{b^2 + c^2 + d + a}$.

34. $\sqrt[3]{3d + ec + a - 1}$.

35. $de + ac\sqrt{ac^2 + 2e + c}$.

36. $(a + c)c$.

37. $(a + e)(c + d)$.

38. $ab(a + b)$.

39. $acd(a + c + d)$.

40. $ac(e - c + b + d)$.

41. $(a + d)^2$.

42. $(e - a)^2$.

43. $(d - a)^3$.

44. $\dfrac{3ad(3a - 2c)^2(e - c)^3}{6ed}$.

45. $\dfrac{ab(c + d)(a + b)}{(e - a)}$.

46. $c^3 - 3c^2a + 3ca^2 - a^3$.

47. If $x = 2$ and $y = 3$, does $3x + 5y = 21$?

48. If $x = 8$, does $7x - 9 = 3x + 25$?

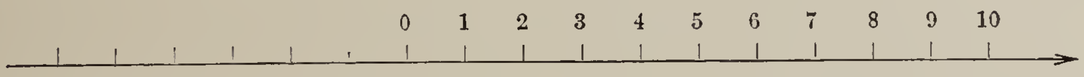
49. Does $x^2 + 5x + 6 = 0$, if $x = 2$? if $x = 3$? if $x = 4$?

50. Does $3x^2 - 14x - 5 = 0$, if $x = 5$? if $x = \frac{1}{3}$? if $x = 6$? if $x = 4$?

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

11. Arithmetical addition and subtraction. Let us suppose that equal distances are taken on a line and the successive points of division marked with the natural numbers as follows:



(*A*)

Such a scale of numbers may be used to illustrate both addition and subtraction as performed in arithmetic.

Thus in adding 5 to 3 we may begin at 3 and count 5 spaces to the right, obtaining the sum 8. We shall obtain the same result if we begin at 5 and count 3 spaces to the right. This process may be stated in general terms thus:

RULE. *To add the number* a *to the number* b, *begin at* b *and count* a *spaces to the right.*

In subtracting 4 from 7 we may begin at 7 and count 4 spaces to the left, thus obtaining 3. This process may be stated in general terms thus:

RULE. *To subtract the number* a *from the number* b, *begin at* b *and count* a *spaces to the left.*

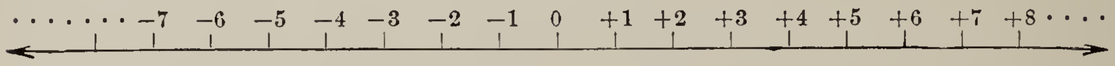
If we attempt to subtract 5 from 4 by the preceding rule, we arrive at the first point of division to the left of zero. Arithmetic has no number to represent such a result; in fact, the subtraction of 5 from 4 is there regarded as impossible. Arithmetically speaking, such a subtraction cannot be performed. We can, however, subtract 4 of the 5 units from the 4 units, leaving 1 unit unsubtracted. Now in algebra it is both convenient and necessary to speak of subtracting a greater number from a less, and to call the portion of the greater number which is unsubtracted the remainder. The fact that such a subtraction is incomplete is indicated by writing a minus sign before the result; thus $4 - 5 = -1$. Hence the first point of division to the left of zero may be thought of as corresponding to -1. Similarly $3 - 5 = -2$; and to -2 may correspond the second point of division to the left of zero.

In like manner $5 - 8 = -3$, which corresponds to the third point to the left of zero. In the same way the fourth point of division to the left of zero would correspond to -4, the fifth point to -5, etc.

Such numbers as -1, -2, -3, etc., are called **negative** numbers. The minus sign is never omitted in writing a negative number, though a letter, as x, may stand for a negative number.

In opposition to negative numbers the ordinary numbers of arithmetic are called **positive** numbers. If a number has no sign before it, or a plus sign, it is a positive number.

The relative order of positive and negative numbers is indicated in the following scale:



(*B*)

EXERCISES

Perform the following additions and subtractions by counting along the preceding scale according to the rules on page 13:

1. Add 4 to 2.
2. Add 4 to -2.
3. Add 5 to -3.
4. Add 3 to -3.
5. Add 2 to -4.
6. Add 5 to -7.
7. Subtract 2 from 5.
8. Subtract 5 from 2.
9. Subtract 6 from 4.
10. Subtract 4 from -2.
11. Subtract 3 from -4.
12. Subtract 2 from -3.

12. Practical use of positive and negative numbers. The scale (*B*) of positive and negative numbers could be used to measure many of the things with which we come in daily contact. In fact, a practical equivalent is in many instances already in use.

Thus in graduating a thermometer a certain position of the mercury is taken as zero, and the degrees are marked both above and below this point. Hence a temperature reading of 18° is indefinite unless accompanied by the words *above zero* or *below zero.* Usually + 18° is taken to indicate the former, while − 18° indicates the latter.

Similarly any point on the earth's equator is in zero latitude. Then latitude 40° N. means 40° north of the equator. In like manner 30° S. means 30° south of the equator. Obviously + 40° and − 30° might be used to convey the same ideas.

EXERCISES

1. If the temperature is now + 10°, what will represent the temperature after a fall (*a*) of 5°? (*b*) of 10°? (*c*) of 18°?

2. If the temperature is now − 12°, what will it be after a rise (*a*) of 7°? (*b*) of 12°? (*c*) of 25°?

3. In the preceding exercise change the word *rise* to *fall* and then answer.

4. A ship sails south from latitude + 13° to latitude − 7°. If one degree is 69 miles, how far did it sail?

5. A ship sails south from latitude + 20° at the rate of 4° daily. In what latitude is it at the end of each of the next 6 days? In how many days will it reach latitude − 16°?

6. A man's property is worth $5200 and his debts amount to $2300. How can positive and negative numbers be used to represent (*a*) each of these amounts? (*b*) the man's financial standing?

Note. The first explanation of positive and negative numbers is, so far as is known, by means of the illustration of assets and debts. This is found in the writings of the Hindus before 700 A.D., long before negative numbers were accepted as having any definite meaning. In the use of this illustration the Hindus were nearly a thousand years in advance of the times.

7. If debts and property be reversed in 6, what would be the answer to (*a*) and (*b*)?

8. The temperature at 6.00 A.M. was $-12°$. During the morning it rose at the rate of 3° an hour. What was the temperature at 9.00 A.M.? 10.00 A.M.? 1.00 P.M.?

13. Addition of positive and negative numbers. As we have seen, subtraction by the use of scale (B) is performed by counting spaces to the left. Now a negative number represents an unperformed subtraction; therefore to add a negative number to another number means to perform this subtraction.

For example, in subtracting 8 from 5, -3 was obtained by beginning at zero and counting 5 to the right and then 8 to the left, leaving a remainder of 3 to the left. Thus when we wish to add -3 to any number, we count 3 spaces to the left from that number. Hence, in general, to *add* a *negative* number n to a given number, begin at the given number and count n spaces to the *left.*

Again, to add -8 to 24 we begin at 24 and count 8 spaces to the left, obtaining 16 as the result; that is,

$$+24+(-8)=+16.$$

Similarly, to add -6 to -4 we begin at -4 and count 6 spaces to the left, obtaining -10 as the result; that is,

$$-4+(-6)=-10.$$

The **numerical** or **absolute** value of a number is its value without regard to sign.

Thus the absolute values of -3, -5, and $+7$ are 3, 5, and 7 respectively. It should be noted that two different algebraic numbers, as $+6$ and -6, may have the same absolute value.

EXERCISES

Perform by the use of scale (B), page 14, the operations:

1. $+3+(+2)$.
2. $-3+(-2)$.
3. $+3+(-2)$.
4. $-3+(+2)$.
5. $7+(-4)$.
6. $-7+(+5)$.
7. $-6+(-4)$.
8. $6+(-5)$.
9. $4+(-7)$.
10. $-3+(-6)$.

The preceding exercises illustrate the correctness of the following working rules:

I. *To add two or more positive numbers, find the arithmetical sum of their absolute values and prefix to this sum the plus sign.*

II. *To add two or more negative numbers, find the arithmetical sum of their absolute values and prefix to this sum the minus sign.*

III. *To add a positive and a negative number, find the difference of their absolute values and prefix to the result the sign of the one which has the greater absolute value.*

The **algebraic sum** of two or more numbers is the number obtained by adding them according to the preceding rules.

The algebraic sum of two numbers is often different from the sum of their numerical values; for example, the algebraic sum of $+9$ and -5 is $+4$, while the sum of their numerical values is 14.

Hereafter the word *add* will mean find the *algebraic sum.*

EXERCISES

Perform the indicated additions:

1. $+4+(+7)$.
2. $-4+(-7)$.
3. $+4+(-7)$.
4. $-4+(+7)$.
5. $+8+(-5)$.
6. $-8+(+5)$.
7. $-12+(-9)$.
8. $-6+6$.
9. $-6+(-6)$.
10. $-4+(+3)+(+6)$.
11. $3+(-7)+(5)+(-4)$.
12. $8+(-2)+(-4)+(+6)$.

Answer the questions asked in the following:

13. $6+?=9$.
14. $6+?=2$.
15. $8+?=12$.
16. $8+?=4$.
17. $-8+?=-10$.
18. $-8+?=-6$.
19. $-10+?=-16$.
20. $-10+?=7$.
21. $12+?=4$.
22. $-12+?=4$.

14. Subtraction of positive and negative numbers. If we wish to subtract 7 from 12, we may do so by answering the question, "What number added to 7 gives 12?" By answering a similar question we can subtract 8 from 15, or 25 from 43, or any number a from another number b. Exercises 13–22, page 17, are therefore exercises in subtraction, for each asks a question similar to the one in the first sentence of this paragraph.

This point of view brings out the relation that the operation of subtraction bears to addition.

EXERCISES

Perform the following subtractions by answering in each case the question, "What number added to the first number gives the second number?"

Subtract:

1. 5 from 8.

2. 9 from 13.

3. 8 from 5.

4. 13 from 9.

5. -5 from -8.

6. -5 from 8.

7. 5 from -10.

8. -6 from -4.

9. 12 from -18.

10. 25 from 13.

11. Change the sign of the subtrahend (if + to −, if − to +) in Exercises 1–10 and then add the subtrahend to the minuend. Are the answers the same as were obtained before?

The results obtained in Exercise 11 illustrate the following important principles:

I. *Subtracting a positive number is the same in effect as adding a negative number of the same absolute value.*

To illustrate: a decrease of $100 in a man's assets is equivalent to an increase of $100 in his liabilities, provided we consider his real financial standing in each case.

II. *Subtracting a negative number is the same in effect as adding a positive number of the same absolute value.*

To illustrate: a decrease of \$75 in a man's liabilities is equivalent to an increase of \$75 in his assets, as far as his net financial standing is concerned.

Hence, for the subtraction of positive and negative numbers, we have the

RULE. *Change the sign of the subtrahend (if it be + to –, if it be – to +) and then find the algebraic sum of the subtrahend (with its sign changed) and the minuend.*

This rule really turns algebraic subtraction into algebraic addition.

EXERCISES

Perform the indicated subtractions:

1. $8-(+5)$.
2. $+8-(-5)$.
3. $+5-(-8)$.
4. $+5-(+8)$.
5. $12-(+9)$.
6. $12-(-9)$.
7. $-12-(9)$.
8. $-12-(-9)$.
9. $+6-(+6)$.
10. $-6-(-6)$.
11. $6-(-6)$.
12. $-6-(+6)$.
13. $-14-(-19)$.
14. $0-(+1)$.
15. $-0-(-1)$.
16. $1-(-2)$.
17. $12-(+3)-(+2)$.
18. $-12-(+3)-(+2)$.
19. $-12-(-3)-(-2)$.
20. $18-(-5)-(7)$.

Answer the questions asked in:

21. $+6+?=10$.
22. $-6+?=-10$.
23. $-3+?=0$.
24. $+6+?=0$.
25. $+6+?=4$.
26. $-8+?=-3$.
27. $-8+?=-5$.
28. $-7+?=7$.
29. $9-?=3$.
30. $-7-?=-5$.
31. $+5-?=-6$.
32. $-7-?=4$.
33. $-5-?=0$.
34. $2-?=0$.
35. $4-?=18$.

Simplify:

36. $12+(3)-(5)$.
37. $12-(-3)+6$.
38. $12-(-4)+(-6)$.
39. $18+(-6)-(+7)$.
40. $-16+(-10)-(+11)$.
41. $-13-(8)+(-14)$.

15. Multiplication of positive and negative numbers. In arithmetic, multiplication was defined as the process of taking one number, the multiplicand, as many times as there are units in another number, the multiplier. The original signification of *times* made this definition meaningless when the multiplier was a fraction; for in $8 \times \frac{3}{5}$, 8 could not be added $\frac{3}{5}$ of a time. The definition was then extended and the product of 8 multiplied by $\frac{3}{5}$ was defined to mean $\frac{8 \times 3}{5}$.

Similarly, since algebra deals with both positive and negative numbers, we must now extend the arithmetical definition of multiplication and **define** what is meant in each of the four cases which follow:

1. $+4 \cdot +3 = ?$
2. $-4 \cdot +3 = ?$
3. $+4 \cdot -3 = ?$
4. $-4 \cdot -3 = ?$

From the arithmetical definition of multiplication $+4 \cdot +3$ means

$$(+4)+(+4)+(+4)=+12;$$

that is,

$$+4 \cdot +3 = +12.$$

Similarly $-4 \cdot +3$ means

$$(-4)+(-4)+(-4)=-12;$$

that is,

$$-4 \cdot +3 = -12.$$

In $+4 \cdot -3$, we mean that 4 is to be subtracted three times. This is the same as subtracting 12 once.

Therefore

$$+4 \cdot -3 = -12.$$

Lastly $-4 \cdot -3$ means that -4 is to be subtracted three times. This is the same as subtracting -12 once, and subtracting -12 once is the same as adding $+12$. Therefore

$$-4 \cdot -3 = +12.$$

Summing up,

$$+4 \cdot +3 = +12.$$
$$-4 \cdot +3 = -12.$$
$$+4 \cdot -3 = -12.$$
$$-4 \cdot -3 = +12.$$

Or, in general terms,

$$
\begin{aligned}
+a \times +b &= +ab.\\
-a \times +b &= -ab.\\
+a \times -b &= -ab.\\
-a \times -b &= +ab.
\end{aligned}
$$

Therefore we have the

LAW. *The product of two numbers having like signs is a positive number, and the product of two numbers having unlike signs is a negative number.*

EXERCISES

Find the products in the following:

1. $+3 \cdot +4.$

2. $+4 \cdot +12.$

3. $-5 \cdot +4.$

4. $+6 \cdot -6.$

5. $-7 \cdot +8.$

6. $-7 \cdot -3.$

7. $-12 \cdot +9.$

8. $+6 \cdot -4.$

9. $-6 \cdot -6.$

10. $+5 \cdot -10.$

11. $+0 \cdot +4.$

12. $-7 \cdot 0.$

13. $+4 \cdot -5 \cdot +6.$

14. $+4 \cdot -5 \cdot -6.$

15. $-4 \cdot -5 \cdot -6.$

16. $12 \cdot +0 \cdot -5.$

17. $9 \cdot -10 \cdot -0.$

18. $-4 \cdot +3 \cdot -6.$

19. $-3 \cdot -2 \cdot -5.$

20. $2 \cdot -3 \cdot +5.$

Note. The famous German mathematician, Leopold Kronecker (1823–1891), once observed that "the good Lord made the positive integers, but man is responsible for all the rest of the numbers." This expresses the truth about numbers as accurately as one can in a single sentence. We count objects from our earliest years, and so use the positive integers naturally. It is only when we come to study mathematics that the necessity for any other kind of numbers is forced upon us. Here we see that negative numbers are a great convenience if we wish to represent the relations between objects where oppositeness in any of its many forms is involved. But the artificial character of negative numbers delayed their intelligent use for many hundred years. To be sure, the Hindus said that "the square of negative is positive," but the statement probably did not mean anything to those who read it. It was not until after the time of Descartes (see page 199) that the rules for operating on negative numbers were understood, even by great mathematicians.

16. Division of positive and negative numbers. When 18 is divided by 9 the result is 2. Here 18 is the dividend, 9 the divisor, and 2 the quotient. The three are connected by the following relation:

$$\text{quotient} \times \text{divisor} = \text{dividend}.$$

We can see that 2 is the correct value of $18 \div 9$, because $2 \times 9 = 18$. This simple test will be applied to determine whether the quotient is a positive or a negative number. All the cases which may arise are represented by the four following questions:

1. $+18 \div +9 = ?$
2. $-18 \div +9 = ?$
3. $+18 \div -9 = ?$
4. $-18 \div -9 = ?$

These questions are answered as follows:

$$+18 \div +9 = +2 \text{ because } +2 \cdot +9 = +18.$$
$$-18 \div +9 = -2 \text{ because } -2 \cdot +9 = -18.$$
$$+18 \div -9 = -2 \text{ because } -2 \cdot -9 = +18.$$
$$-18 \div -9 = +2 \text{ because } +2 \cdot -9 = -18.$$

In 1 and 4 the dividend and divisor have *like* signs and the sign of the quotient is *plus.* In 2 and 3 the dividend and divisor have *unlike* signs and the quotient is *minus.*

Therefore we have the

RULE. *The quotient of two numbers having like signs is a positive number, and the quotient of two numbers having unlike signs is a negative number.*

The result of multiplication by zero is given a definite meaning in arithmetic and algebra, namely zero; but in both subjects *division by zero is always excluded.* If zero were used as a divisor, numerous contradictions would arise of which the following is an illustration:

Obviously $0 \cdot 4 = 0,$
and $0 \cdot 6 = 0.$
Therefore $0 \cdot 4 = 0 \cdot 6.$
Dividing each by zero, $4 = 6.$

Note. The Hindus were the first to express the laws that govern the operations with the number 0. In fact, they were the first to have such a symbol. In the twelfth century a Hindu writer states that $a + 0 = a$, that $\sqrt{0} = 0$, and that $0^2 = 0$. Of course he did not express himself in terms of these symbols, but in the notation of his time.

EXERCISES

Perform the indicated division:

1. $+10 \div +2$.
2. $-10 \div -2$.
3. $-15 \div +3$.
4. $+14 \div -7$.
5. $-18 \div -2$.
6. $-7 \div +7$.
7. $0 \div +5$.
8. $0 \div -5$.
9. $+18 \div +3 \div -2$.
10. $+45 \div -5 \div -3$.
11. $-64 \div +8 \div -2$.
12. $+96 \div -6 \div +4$.
13. $72 \div +9 \div -4$.
14. $60 \div -5 \div -12$.
15. $48 \div +3 \div -4$.
16. $\frac{-10}{?} = 2$.
17. $\frac{-12}{?} = -2$.
18. $\frac{+16}{?} = 8$.

If the first of several numbers connected by either plus or minus signs is a positive number, its sign is omitted; thus $+4 - 3 + 6$ is written $4 - 3 + 6$. If the sign of the 4 had been negative, its sign could not have been omitted.

If each of two or more numbers be inclosed in a parenthesis with no sign of operation connecting them, the sign of multiplication is always understood; thus $(6)(3)$, or even $6(3)$ or $(6)3$ means $6 \cdot 3$.

EXERCISES

Simplify the following:

1. $(7)+(5)$.
2. $(7)-(5)$.
3. $(7)+(-5)$.
4. $(7)-(-5)$.
5. $(-7)+(5)$.
6. $-7+5$.
7. $(-9)-(4)$.
8. $-9-4$.
9. $-11+(-13)$.
10. $-6-(-10)$.
11. $-6+10$.
12. $8+(-10)$.

13. $12 - 18.$

14. $-18 - 12.$

15. $15 - 14.$

16. $+7 - 0.$

17. $-0 - 3.$

18. $(-3)(6).$

19. $(-5)6.$

20. $(7)(-5).$

21. $8(-3).$

22. $(-5)(-12).$

23. $-3(-8).$

24. $-5 \cdot 4.$

25. $5 \cdot 0.$

26. $0 \cdot (-9).$

27. $4 \cdot 8.$

28. $-3 \cdot 6.$

29. $12 \div (-2).$

30. $-12 \div 2.$

31. $-39 \div (-3).$

32. $45 \div (-15).$

33. $0 \div (-6).$

34. $0 \div 3.$

35. $-27 \div 9.$

36. $3 - 5 + 6.$

37. $-4 + 6 - 2 + 1.$

38. $-4 + 6 + 2 - 1.$

39. $2 - 3 + 4 - 5 - 6.$

Add:

40. $\begin{array}{r} 7 \\ -2 \\ 3 \\ -5 \\ \hline \end{array}$

41. $\begin{array}{r} 6 \\ -2 \\ -3 \\ 4 \\ \hline \end{array}$

42. $\begin{array}{r} -8 \\ 6 \\ 2 \\ -5 \\ \hline \end{array}$

43. $\begin{array}{r} 4 \\ -9 \\ -3 \\ 6 \\ \hline \end{array}$

Simplify:

44. $3 \cdot 6 \div 3.$

45. $-4(7) \div (-2).$

46. $3(-6) \div 2.$

47. $4 \cdot 6(-8) \div (-16).$

48. $18 \div (-3) \cdot 6 \div 4.$

49. $3^2.$

50. $(-3)^2.$

51. $2^3.$

52. $(-2)^3.$

53. $(-4)^3 + (4)^2.$

54. $(-1)^2 + (-1)^3 + (-2)^2 + (-2)^3.$

55. $6 + 3 \cdot 2 + 18 \div (-3).$

56. $5^2 - 4 \div (-2) + 6(-3).$

57. $3 \cdot 6 \div 9 - 2 \cdot 6 \div 4 + (-3)^2.$

58. $6 + 6 \cdot 3^2 - 5^2 \cdot 2 \div 10.$

If $x = 3$ and $y = -2$, find the value of:

59. $y^2.$

60. $y^3.$

61. $y^4.$

62. $y^5.$

63. $2y^2.$

64. $2y^3.$

65. $5x^2y^2.$

66. $4x^2y^4.$

67. $x^2 - y^2$.

68. $x^3 - y^3$.

69. $x^2 + 2xy + y^2$.

70. $x^2 - 2xy + y^2$.

71. $(x + y)(x - y)$.

72. $x^3 + 3x^2y + 3xy^2 + y^3$.

73. $y^3 - 3xy^2 + 3x^2y - x^3$.

74. Does $4x - 2 = 2x + 8$, if $x = 5$?

75. Does $3x - 5 = 2x + 8$, if $x = -9$?

76. Does $x^2 - x - 12 = 0$, if $x = 4$? if $x = -8$? if $x = -4$?

77. Does $3x^2 + 19x = 14$, if $x = \frac{2}{3}$? if $x = 2$? if $x = -7$?

78. At 7.00 A.M. on a certain day the thermometer registered 15 degrees above zero. The mercury then fell at the rate of 3 degrees per hour. What was the temperature at 9.00 A.M.? at noon? at 3.00 P.M.?

79. With reference to a certain assumed level, a surveyor found the heights of 5 points to be + 30 feet, − 7 feet, + 18 feet, − 10 feet, and + 16 feet respectively. What was the average height of the 5 points? What meaning has the result?

80. A sixth point whose height was − 38 feet was later included in the preceding survey. Find the mean height of the 6 points. What meaning has the result?

81. Later a seventh point was added to the preceding survey. The average height of the 7 points was then zero. Find the height of this last point.

82. Euclid lived about 300 B.C. Sir Isaac Newton died in 1727 A.D. If dates before Christ are considered negative and those after Christ be considered positive, how might these dates be written?

83. What is the meaning of the date − 450? of + 1910? What is the difference in time between these two dates?

84. In still water a gasoline launch can travel 8 miles per hour. Using positive and negative numbers, represent its rate both up and down a river which runs $1\frac{1}{2}$ miles per hour.

85. A boat is traveling 12 miles per hour. A man on its deck is walking 3 miles per hour. Using positive and negative numbers, represent the rate at which he approaches his destination when he walks toward the bow and when he walks toward the stern.

86. A balloon capable of supporting 500 pounds is held down by 10 men whose average weight is 150 pounds. Using positive and negative numbers, represent the weight of the balloon, the men, and the balloon and men together.

87. A man swims in still water at the rate of $1\frac{1}{2}$ miles per hour. If he swims in a river which flows 2 miles per hour, represent his rate (*a*) when he swims downstream; (*b*) when he swims upstream. What is the practical meaning of the last answer?

CHAPTER III

ADDITION

17. Addition of monomials. A number symbol consisting of a numeral, or a letter, or a product of letters alone, or the product of a numeral and one or more letters is called a **monomial** or **term.**

Thus 5, $-a$, b^4, a^2x, $-4cy^2$, $\frac{3}{5}a^2x^3y$, x^a, and x^{a+2} are terms. Frequently, where no confusion would arise, expressions like $(a+b)$, $3(x-y)$, $5\sqrt{x^3}$, and $\sqrt{a-x}$ are also called terms, for often they may be replaced by a single letter.

The **literal** part of a term is the portion composed of letters.

Similar terms are integers and fractions, like numerical roots, or such terms as have like literal parts.

Thus 3, -7, and 9 are similar terms as well as $\sqrt{2}$ and $3\sqrt{2}$. Also a, $4a$, and $-10a$ are similar terms as are a^2x, $-3a^2x$, and $7a^2x$.

Dissimilar terms are unlike numerical roots or such terms as have unlike literal parts.

Thus 4, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt[3]{5}$ are dissimilar terms as well as $3a$, $4b$, and $6c^2$. Also $7a^2x$, $3ax^2$, and $5ax$ are dissimilar terms.

We know that 6 acres and 3 acres = 9 acres. Similarly $6a+3a=9a$, and $6a+(-3a)=3a$, and $5xy+6xy=11xy$. In like manner the sum of $8y$, $-3y$, $2y$, and $-y$ is $6y$. Such terms as $-y$, x, ay, and $-c^2x$ are equivalent to $-1y$, $+1x$, $+1ay$, and $-1c^2x$, the coefficient 1 being always understood if no numerical coefficient is written.

Thus for adding similar terms we have the

RULE. *Find the algebraic sum of the numerical coefficients and prefix this result to the common literal part.*

EXERCISES

Find the algebraic sum of:

1. $18 - 5 + 6$.

2. $18a - 5a + 6a$.

3. $12 - 7 + 3 + 4$.

4. $12a^2 - 7a^2 + 3a^2 + 4a^2$.

5. $6 - 4 - 17 + 20$.

6. $6ab - 4ab - 17ab + 20ab$.

7. $15 - 17 + 8 - 12 - 25$.

8. $8abc - 17abc - 4abc + 15abc - 12abc$.

9. $11 + 5 - 9 - 3 + 16 - 25$.

10. $16ac - 9ac + 5ac - 2ac - 3ac + 11ac$.

11. $7x + 4x - 15x - 8x + 3x$.

12. $12y - 17y + 10y + 20y - 25y$.

13. $4xy - 8xy - 12xy + 13xy - xy$.

14. $14x^2 - 13x^2 + x^2 - 5x^2 + 4x^2$.

15. $5y^2 - 2y^2 - 11y^2 + y^2 - 7y^2$.

16. $7a^2b - 5a^2b + 8a^2b - a^2b + 9a^2b$.

Obviously $2 + 3 = 3 + 2$, and $2 - 3 + 5 = 2 + 5 - 3 = -3 + 5 + 2$, etc. This illustrates the law that in addition the terms may be arranged in any order. Hence $6d + 7c = 7c + 6d$, and the sum of 3 and x is either $x + 3$ or $3 + x$; also $a + b = b + a$, and $a + b + c = b + c + a = c + a + b$, etc.

Algebraic expressions for numbers with unlike literal parts, such as $6d$ and $7c$, may be added by writing them one after the other with a plus sign between them; thus $6d + 7c$. The addition of $6d$ and $-7c$ is indicated by writing $6d + (-7c)$, which is the same as $6d - 7c$. Similarly the sum of $3x$, $-2y$, and $-7z$ may be written $3x + (-2y) + (-7z)$, or, more simply, $3x - 2y - 7z$.

Thus for adding dissimilar terms we have the

RULE. *Write the terms one after another in any order, giving to each its proper sign.*

If similar and dissimilar terms are to be added, the two preceding rules must be observed.

EXERCISES

Find the algebraic sum of:

1. a, $3b$, and $-c$.
2. $4x$, $-2b$, $3y$, and 10.
3. $3ab^2$, $2bx$, $-cy$, and $4a^2b$.
4. $5x^3y$, $-5xy^3$, c^3y, and $-2cy^3$.
5. $4x$, $-3a$, $2b$, $-5x$, and $3y$.
6. $5a$, $-4b$, $+3c$, $6b$, and $-2c^2$.
7. $3a^2$, $+2b^2$, $-5c^2$, $-4b^2$, and $5a^2$.
8. $4a^3b$, $-4ab^3$, $-3a^2b$, $3ab^2$, $4a^3b$, and $2ab^3$.
9. $-4a^2b+6a^2b^2-15a^2b^2+3a^2b^2+0a^2b^2$.
10. $-b^3-23b^3+17b^3+b^3-0b^3+13b^3$.
11. $12a^3b+6a^3b-a^3b+16a^3b-13a^3b-25a^3b$.
12. $11\sqrt{a}-14\sqrt{a}+21\sqrt{a}-\sqrt{a}$.
13. $3\sqrt{x-y}-\sqrt{x-y}+9\sqrt{x-y}-7\sqrt{x-y}$.
14. $3(a+b)-2(a+b)+8(a+b)$.
15. $-7(a-2b)$, $(a-2b)$, $12(a-2b)$.
16. $8(a+b)+3(a-b)-4(a+b)-2(a-b)$.
17. $4(x-y)-3(x+3)-6(x-y)+5(x-3)$.
18. $(2x-y)^2-3(2x-y)^2+4(2x-y)^2-7(2x-y)^2$.

18. Addition of polynomials. A **polynomial** is an algebraic expression consisting of two or more terms.

It is not usual to call an expression a polynomial if any of its terms contain a letter under a radical sign. Thus we shall not call expressions like $\sqrt{x-3}+4$ polynomials.

A **binomial** is a polynomial of two terms.

A **trinomial** is a polynomial of three terms.

EXAMPLE

Add the following polynomials: $4a - 6b - a^2c$; $3b + 4a^2c$; $-3a - 7a^2c + 10$; $5a + 3b - 6$.

Solution:

$$\begin{array}{lrrrr}
 & 4a & -6b & -a^2c & \\
 & & 3b & +4a^2c & \\
 & -3a & & -7a^2c & +10 \\
 & 5a & +3b & & -6 \\
\hline
\textit{Sum,} & 6a & & -4a^2c & +4
\end{array}$$

For the addition of polynomials we have the

Rule. *Write similar terms in the same column.*

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

A **check** on an operation is another operation which tests the correctness of the first.

For example, in arithmetic the result of division is checked by multiplication; thus the check for $132 \div 6 = 22$, is $22 \cdot 6 = 132$.

In the following example the letters a, b, and c represent any numbers whatever. Therefore in order to check the result we may give them any numerical values we please. Let $a = 1$, $b = 1$, and $c = 1$.

EXAMPLE

Add the polynomials $5a - 3b - 3c$, $2a - 5b + 6c$, and $-3a + 2b - 4c$.

$$\begin{array}{lrcr}
 & \textbf{Solution:} & & \textbf{Check:} \\
 & 5a - 3b - 3c & = & 5 - 3 - 3 = -1 \\
 & 2a - 5b + 6c & = & 2 - 5 + 6 = 3 \\
 & -3a + 2b - 4c & = & -3 + 2 - 4 = -5 \\
\hline
\textit{Sum,} & 4a - 6b - c & & -3 \\
\text{But} & 4a - 6b - c & = & 4 - 6 - 1 = -3
\end{array}$$

We conclude that the addition is correctly performed, since the numerical value of the sum is -3 and the sum of the numerical values of the three polynomials to be added is -3 also.

A numerical check will usually detect errors, though not always. Two errors may be made, one of which offsets the other; these errors would not be detected by a numerical check like the preceding one. Thus if in the preceding exercise the incorrect sum $4a - 5b - 2c$ had been obtained, the substitution of 1 for a, b, and c in this result would have given -3, an apparent check.

The number of times that errors will thus balance one another, however, is small compared to the total number of errors made; hence the check illustrated is practically very useful. If a more reliable check is desired in similar exercises, it can be obtained by the substitution of a different number for every letter.

EXERCISES

Add the following polynomials and check the results:

1. $x + y + z$, $x - 2y + 3z$, and $3x + 4y - 7z$.

2. $2x + 5y - z$, $3x - 8y + 6z$, and $x - y - z$.

3. $3x + 5y$, $4x - 7y + 6z$, and $3x - 3y - 3z$.

4. $7x - y + 3z$, $5x - 4z$, and $2x + 6y - 5z$.

5. $4x - 3y - 5z$, $6y - 2z$, and $7x - 6y - 4z$.

6. $x + 2z + 3y$, $y - 3z + x$, and $z - 2x - 4y$.

7. $5x - y + 3z$, $2y - 11z + x$, and $9z - 7y$.

8. $8a - 7b - 6c$, $5c - 4a - 3b$, and $3b + 7c$.

9. $9ac - bc$, $8ab - 4ac$, and $-12ab - ac$.

10. $a^2 - 4a + 10$, $5a - 6a^2 + 4$, and $3a - 16 + 2a^2$.

11. $9 - a^2 + a$, $-7 + 6a^2 - 4a$, and $5a^2 - a$.

12. $\frac{1}{2}x - \frac{2}{3}y + \frac{2}{5}z$, $x - y + 2z$, and $\frac{1}{3}x - \frac{1}{2}y + \frac{1}{10}z$.

13. $\frac{1}{4}x - \frac{3}{20}z$, $\frac{1}{8}x + \frac{4}{9}y$, and $\frac{3}{5}z - \frac{5}{3}y$.

14. $a - 3(x - y) + z$, $5 - 10a + 4(x - y)$, and $-2(x - y) + 6$.

15. $a + d + 2(b - c)$, $7(b - c) + 6d - 12a$, and $11a - 5(b - c)$.

Combine similar terms in the following polynomials:

16. $7a^2 - 13b^2 + 12c^2 + 15b^2 - a^2 - 7c^2 + 3b^2 + 5c^2$.

17. $x^2 - 2xy + y^2 - 4xy - 4x^2 - y^2 - 16x^2 - 8xy + y^2 + y^3$.

18. $5ab - a^2 + b^2 - 4ab - 9b^2 + 5a^2 - 2ab + 2b^2$.

19. $3x^2 - 6x + 11 - 4x - 8 - 3x^2 + 5x - 16 + 13x$.

20. $12c^2 - 10bc + 8b^2 - bc - 6b^2 - c^2 + c^3 - 11c^2$.

21. $4x^2y - xy + y^2 - 3xy + 4xy^2 - 2x^2y + 4y^2 - 3xy + x^2y$.

22. $\frac{1}{2}a + \frac{1}{3}b - \frac{1}{4}c - c + \frac{1}{2}b - \frac{2}{3}a - b + \frac{1}{2}c - \frac{3}{4}a + 7$.

The sum of $5x$ and $2x$ may be written $(5 + 2)x$. This is not usual or necessary, as 5 and 2 can be combined and the result written $7x$. In adding $5x$ and ax, however, the 5 and the a cannot be combined and the result expressed by a single character, so the sum is written $(5 + a)x$. Similarly $ax - 3x = (a - 3)x$, $ax + bx = (a + b)x$, and $ax - bx + x = (a - b + 1)x$.

Write the following with polynomial coefficients:

23. $ax + bx + cx$.

24. $2ax - 3x + bx$.

25. $3ax - 4cx + x$.

26. $3ax^2 - bx^2 - x^2 + a^2x^2$.

27. $by - 4cy - y - 4by$.

28. $a(b + c) + 3(b + c)$.

29. $4(a - x) - 3b(a - x)$.

30. $8a(a + 3b) - 1(a + 3b)$.

31. $7b(x^2 + y^2) - a(x^2 + y^2) + (x^2 + y^2)$.

CHAPTER IV

SIMPLE EQUATIONS

19. Definitions and axioms. An **equation** is a statement of equality between two equal numbers or number symbols.

Thus $2 = 5 - 3$, $a - 2b = 3a + b - 2a - 3b$, $4x = x + 12$, and $x^2 - 5x + 6 = 0$ are equations.

The part of an equation on the left of the equality sign is called the *first* or *left member*, that on the right, the second or right member.

In an equation a letter whose value is sought is called the *unknown letter*, or simply the *unknown*.

An **axiom** is an evident truth which is accepted without proof.

In the solution of equations constant use is made of four axioms.

AXIOM I. *Adding the same number to each member of an equation does not destroy the equality.*

AXIOM II. *Subtracting the same number from each member of an equation does not destroy the equality.*

AXIOM III. *Multiplying each member of an equation by the same number does not destroy the equality.*

AXIOM IV. *Dividing each member of an equation by the same number (not zero) does not destroy the equality.*

If an equation is in a form as simple as $3x = 12$, it can easily be solved by dividing each member by the coefficient of x.

Thus dividing each member of $3x = 12$ by 3, the coefficient of x, we get $x = 4$.

If all terms containing the unknown letter are in one member and all numerical terms in the other, the like terms may be united and the equation solved as before.

Thus $5x - 2x + x = 8 + 15 - 3$ becomes, when like terms are united, $4x = 20$, and dividing each member by 4, we obtain $x = 5$.

Usually numerical terms as well as terms containing the unknown letter will be found in each member of an equation, as in $5x - 3 = 2x + 18$. By the use of one or more of the preceding axioms it is always possible to change the form of such equations until they are similar to the equation $3x = 12$, which, as we have seen, can easily be solved.

A **simple** or **linear** equation is one that may be put in a form in which:

(*a*) There is at least one unknown.
(*b*) The exponent of each unknown is 1.
(*c*) No term contains more than one unknown.
(*d*) No unknown occurs in any exponent.
(*e*) No unknown occurs in any denominator.

Thus $5x - 2 = 8$, $4x = y - 18$, $4n - 2 = 3n + 8$, and $x - 2y + 3 = 6$ are simple equations, while $2^x = 4$, $x - xy = 3$, and $\frac{1}{x} + \frac{2}{x+1} = 2$ are not.

EXAMPLE

Solve $5x - 4 = 2x + 17$.

Solution: $5x - 4 = 2x + 17$.

Subtracting $2x$ from each member,

$3x - 4 = 17$. Ax. II

Adding 4 to each member, $3x = 21$. Ax. I

Dividing each member by 3, $x = 7$. Ax. IV

Checking the solution of an equation is often called **testing**, or **verifying**, the result. For this we have the

RULE. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify the result until the two members are seen to be identical.*

Check: $5x - 4 = 2x + 17$.

Substituting 7 for x, $5 \cdot 7 - 4 = 2 \cdot 7 + 17$.

Simplifying, $35 - 4 = 14 + 17$,

or $31 = 31$.

EXERCISES

Find the value of the unknown in the following equations and verify results:

1. $x + 5 = 11.$

2. $x - 4 = 12.$

3. $3x + 10 = 28.$

4. $5x - 6 = 19.$

5. $9x - 12 = 6.$

6. $4x = 12 + x.$

7. $6x = 20 + 2x.$

8. $9n = 40 - n.$

9. $13n = -5n + 36.$

10. $-4n = -13n + 27.$

11. $3y + 2 = y + 8.$

12. $5 + 4y = 3y + 20.$

13. $2y - 3 = 17 - y.$

14. $8x - 15 = 6x - 15.$

15. $-7h + 19 = 25 - 9h.$

16. $-7x + 18 = 4x + 18.$

17. $5k - 4 = 3k + 18.$

18. $8 - 6x + 12 + 10x = 26.$

19. $x + 2x + 18 + x + 2x + 18 = 116.$

20. $4x - 3 + 8x - 17 = 40 + 6x - 54.$

EXERCISES

1. A rectangle is three times as long as it is wide. If x represents the width in feet, what will represent the length? the perimeter?

2. A rectangle is 10 feet longer than it is wide. If x represents the width in feet, what will represent the length? the perimeter?

3. A rectangle is 18 feet longer than it is wide. If x represents the length in feet, what will represent the width? the perimeter?

4. The length of a certain rectangle is 4 feet more than twice the width. Represent the width, the length, and the perimeter in terms of one letter and numbers.

5. The numbers 3, 4, 5, 6, etc., are consecutive integers. How much greater is each than the preceding one?

6. If n represents an integer, what will conveniently represent the next consecutive one?

7. If n represents an integer, represent the next two consecutive integers. What will represent the sum of these two?

8. If n represents the first of four consecutive integers, what will represent the other three? the sum of the four?

9. The numbers 3, 5, 7, 9, 11, etc., are consecutive odd integers. How much greater is each than the preceding one?

10. If n is any odd number, what will represent the next greater odd number? What will represent the odd number preceding n?

11. Represent three consecutive odd numbers of which n is the first. What will represent the sum of the three?

12. Represent four consecutive even numbers. Find their sum.

13. Represent three numbers of which the second is twice the first, and the third three times the second.

14. Represent three numbers of which the second is 10 more than the first, and the third 7 less than the second.

15. If a man's age now is represented by x, what will represent his age 4 years ago? 6 years hence?

16. A's age is twice B's. Represent the age of each in terms of x: (*a*) now; (*b*) 7 years ago; (*c*) 12 years hence.

Express the following statements as equations:

17. The sum of 8 and x is 5.

18. x is 2 less than 10.

19. x is 3 greater than 5.

20. x is 5 more than y.

21. Three times x is 21.

22. Four times a is greater than 18 by 2.

23. Eight added to x gives the same result as x taken from 34.

24. Nine subtracted from $2x$ gives the same result as 14 added to x.

25. Twice x added to three times x is 48 more than x.

26. Twelve taken from three times x gives the same result as x added to 50.

PROBLEMS

1. One number is twice another, and the sum of the two is 135. Find both numbers.

2. One number is four times another, and the sum of the two is 105. Find both numbers.

3. One number is five times another, and the difference of the two is 52. Find both numbers.

4. One number is 5 greater than another, and their sum is 129. Find both numbers.

5. One number is 18 greater than another, and their sum is 168. Find both numbers.

6. A rectangle is five times as long as it is wide, and its perimeter is 156 feet. Find the length and the width.

7. A rectangle is 12 feet longer than it is wide, and its perimeter is 96 feet. Find the length and the width.

8. The perimeter of a rectangle is 98 feet, and its length is 4 feet more than twice the width. Find the length and the width.

9. Find the dimensions of a rectangle whose perimeter is 88 feet, and whose length is 20 feet less than three times the width.

10. Find three consecutive numbers whose sum is 45.

11. Find four consecutive numbers whose sum is 106.

12. Find five consecutive numbers whose sum is 85.

13. Find three consecutive odd numbers whose sum is 291.

14. Find three consecutive even numbers whose sum is 66.

15. Find five consecutive odd numbers whose sum is 315.

16. A's age is two years less than B's, and the sum of their ages is 43 years. Find the age of each.

17. B's age is twice A's. Six years from now the sum of their ages will be 54 years. How old is each now?

18. The United States has 52,000 more miles of railway than Europe, and together they have 402,000 miles. Find the mileage of each.

19. The Nile is 500 miles longer than the Amazon and 300 miles shorter than the Mississippi (entire). The sum of the lengths of the three is 11,500 miles. Find the length of each.

20. Mount McKinley is 6313 feet higher than Pike's Peak. The sum of the heights of the two mountains is 34,607 feet, which is 5605 feet more than the height of Mount Everest. Find the height of each.

21. The combined horse power of a Mallet Compound freight engine (Erie R. R.), of a Pacific passenger engine (Pennsylvania R. R.), and of a Baltimore and Ohio electric tractor is 11,200. The horse power of the freight engine is 1800 more than that of the electric tractor and 1000 less than that of the passenger engine. Find the horse power of each.

Note. The process of developing a simple and clear means of expressing an equation has been a slow one. The very first writer on mathematics of whom we know anything, an Egyptian priest named Ahmes, who lived nearly two thousand years before Christ, called the unknown *heap* instead of x. One of his problems is as follows: "Heap; its seventh, its whole, it makes nineteen." This we should express by the equation $\frac{x}{7} + x = 19$. The Hindus often used the word *color* for the unknown, and if there was more than one unknown in the equation, the names of different colors would be used; thus they might express the product xy by "black times yellow." The Arabs used the word *root* with a similar meaning, and to this day we call the result of solving an equation its root (see page 44). The early European mathematicians usually called the unknown *res*, the Latin word for *thing*, and it was not until after the time of Vieta (see page 257) that the unknown was regularly denoted by a letter. The use of x for this purpose originated with Descartes in 1637.

CHAPTER V

SUBTRACTION

20. Subtraction of monomials. The principles stated on page 18 apply to the subtraction of monomials as well as to the positive and negative numbers there used.

For subtracting one monomial from another we have the

RULE. *Change the sign of the subtrahend; then find the algebraic sum of this result and the minuend.*

EXAMPLES

1. From $+8a$ take $+3a$.

Solution: $+8a$ minus $+3a = 8a - 3a = 5a$.

2. From $6ax$ take $-4ax$.

Solution: $6ax$ minus $-4ax = 6ax + 4ax = 10ax$.

3. Subtract $7a^2b$ from $-10a^2b$.

Solution: $-10a^2b$ minus $7a^2b = -10a^2b - 7a^2b = -17a^2b$.

4. Subtract $-9ay^3$ from $3ay^3$.

Solution: $3ay^3$ minus $-9ay^3 = 3ay^3 + 9ay^3 = 12ay^3$.

The difference of two dissimilar monomials cannot be written as a single term, but is expressed by a binomial, as follows:

5. Subtract $+a$ from $+b$.

Solution: $+b$ minus $+a = b - a$.

6. Subtract $-5b$ from $3c$.

Solution: $3c$ minus $-5b = 3c + 5b$.

7. Subtract $4xy$ from $-3x^2z$.

Solution: $-3x^2z$ minus $4xy = -3x^2z - 4xy$.

As soon as possible the student should learn to change the sign of the subtrahend *mentally*.

EXERCISES

Subtract the first monomial from the second, and also the second monomial from the first, in each of the following:

1. $2x$, $3x$.

2. $4x$, $3x$.

3. $-2x$, $-3x$.

4. $-5x$, $-3x$.

5. $-x$, $4x$.

6. $-x$, $-3x$.

7. $-c$, $5c$.

8. $-ac$, $-5ac$.

9. $8a^2c$, $-11a^2c$.

10. $6x^2y^2$, $-6x^2y^2$.

11. $3x$, 0.

12. $-4ab$, 0.

13. a, b.

14. c, $2x$.

15. x, $-4y$.

16. $-3a$, $2b$.

17. $-2a$, $-5b$.

18. $-2a$, $-2a$.

21. Subtraction of polynomials. For the subtraction of polynomials we have the

RULE. *Write the subtrahend under the minuend so that similar terms are in the same column.*

Change the sign of each term of the subtrahend.

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

EXAMPLE

Subtract $5x - 2y - 7z + 2$ from $3x + 8y - 5z$, and check the result by letting $x = 1$, $y = 1$, and $z = 1$.

Solution and Check:

$$\begin{array}{rrrrrl}
 & 3x + & 8y - 5z & & = 3 + 8 - 5 & = 6 \\
 & 5x - & 2y - 7z + 2 & & = 5 - 2 - 7 + 2 & = \underline{-2} \\
\hline
\textit{Difference,} & -2x + & 10y + 2z - 2 & = & & 8 \\
 & -2 & +10 \quad +2 \quad -2 & = 8 \text{ also.} & &
\end{array}$$

We might also apply the check:

difference + subtrahend = minuend.

EXERCISES

Subtract the first number from the second, and also the second number from the first, in Exercises 1–9:

1. $a + 2$, $a + 3$.

2. $a - 4$, $a - 2$.

3. $2a - 3$, $3a + 5$.

4. $3a + 7$, $9 - 4a$.

5. $4a$, $4a - 3$.

6. $2ab - 5c$, $5c$.

7. $3xy - a$, $5xy$.

8. 0, $x + 3$.

9. $2x - 3$, 0.

Subtract the first polynomial from the second in Exercises 10–17, and check the work numerically:

10. $x - 2y + 3z,\ 2x - 2y + z.$

11. $4x - 8y - 2z,\ 4x - 5y + 3z.$

12. $3a - 2b,\ 4a - b + 2c.$

13. $5a - 4b - 3c,\ 3a - 5b.$

14. $a - x - y,\ b - x + y.$

15. $3a - 2b - c + 6,\ 4a - b + 5c.$

16. $2a - 2b - 2c + 4,\ a - 3b.$

17. $a + b - c,\ c - d + e.$

Find the expression which added to the first will give the second in:

18. $x - 2y + z,\ 2x + 5y - 3z.$

19. $7x - 9y + 3z,\ 5x - 2y - 4z.$

20. $5a - 4b - 6c,\ 6a + 3b.$

21. $a + b - 2c,\ 5.$

22. $3ab + c,\ 3xy + z.$

23. $2x - 4y - z,\ 0.$

24. $2x^2y - 3xy^2,\ y^2x + yx^2 + z.$

Find the expression which subtracted from the first will give the second in Exercises 25–30:

25. $x - 2y + z,\ 3x + 2y - z.$

26. $x^2 - 7x + 10,\ 14x - 8 + 3x^2.$

27. $x - y + z,\ 5x + 3y - 8z.$

28. $4 - 8x^2,\ x^2 - 5x + 6.$

29. $3a - 5b + c,\ 0.$

30. $4a - 6b + 8y,\ x - 12.$

31. Subtract the sum of $a^2 - 2ab + b^2$ and $a^2 - 12ab + 20$ from $a^2 - 13a + 30.$

32. Subtract the sum of $a - 3b + c$ and $4a + 5b - 6c + 4$ from $a - b + c - x$.

33. From the sum of $5x + 3x^2y - 15xy^2$ and $-6x - 12y^2x + 7yx^2$ subtract $11x - 5x^2y + 7y^2x$.

34. From the sum of $4abc^2 - 3ab^2c + 2a^2bc$ and $6ac^2b - 5acb^2 - 4a^2b^2c$ subtract $2c^2ab - 3a^2bc + 7acb^2$.

35. From the sum of $3x - 4xy - 2z$ and $7xy - 4z - 3x$ take the sum of $5z - 2xy - a^2bc$ and $9x - 6ba^2c - z$.

36. Simplify $(4x - 3y + 6) + (3x + 5y - 10)$.

37. Simplify $(7c + 5d - e) - (4c + 5d - 9e)$.

38. Simplify $(x^2 + 2x + 5) + (2x^2 + x - 10) - (x^2 - 5x + 3)$.

39. Simplify $(x + 3y - 2z) + (4x - 5y + 3z) - (3x - 2y - 6z)$.

Find the algebraic sum of:

40. $(5x + 3y - z) + (4y + 7z) - (x - y + 3z)$.

41. $4x - 3y + 7 - (2x - 5y - 4) + (4x - 8)$.

42. $3c - 5d - e - (5 + 6d + 11e) - (5c + 4e)$.

43. $3a + 3b - 4c - (-3b - 3c - 4) - (4a + x - 8c)$.

CHAPTER VI

IDENTITIES AND EQUATIONS OF CONDITION

22. Kinds of equations. Equations are of two kinds, — **identities** and **equations of condition.**

An **identity** is an equation in which, if the indicated operations are performed, the two members become precisely alike, term for term.

Thus $4 \cdot 5 + 3 \cdot 4 = 8 \cdot 5 - \frac{4 \cdot 6}{3}$ is an identity, for, performing the indicated operations, it becomes $20 + 12 = 40 - 8$, or $32 = 32$.

Similarly the equation $2a + 3b - 4 = 3a - 2b + (5b - a - 4)$ is an identity, for, performing the indicated addition in the second member, it becomes $2a + 3b - 4 = 2a + 3b - 4$, in which the two members are alike, term for term.

A literal identity is true for *any* numerical values of the letters in it.

Thus the literal identity $(a + 3)^2 = a^2 + 6a + 9$ becomes, when $a = 5$, $(5 + 3)^2 = 5^2 + 6 \cdot 5 + 9$, or $8^2 = 25 + 30 + 9$, or $64 = 64$. If a is zero, the identity becomes $(0 + 3)^2 = 0 + 6 \cdot 0 + 9$, or $9 = 9$. If a is 2, we obtain $(2 + 3)^2 = 2^2 + 6 \cdot 2 + 9$, or $25 = 25$.

Similarly $(a + b)^2 = a^2 + 2ab + b^2$ is an identity, and is true for all values of a and b. If $a = 2$ and $b = 3$, this identity becomes $25 = 25$. If $a = -3$ and $b = 5$, it becomes $4 = 4$. If $a = 0$ and $b = 3$, it becomes $9 = 9$. If $a = -4$ and $b = 5$, it becomes $1 = 1$. In this way the literal identity becomes a numerical identity for any numerical values of a and b.

An equation which is true only for certain values of a letter in it, or for certain sets of related values of two or more of its letters, is an **equation of condition,** or simply an equation.

Every equation of condition may be regarded as asking a question. Thus the equation $3x + 2 = 15$ asks, "What number when multiplied by 3 and the product increased by 2 gives 15 as the result?"

The equations used in solving the problems on pages 5–7 are equations of condition. The condition there expressed in ordinary language in the problem was translated into the algebraic language of the equation.

The equation $4x = x + 12$ is true only when $x = 4$. If 4 is put for x, the equation becomes the identity $4 \cdot 4 = 4 + 12$, or $16 = 16$. Clearly the result is false if 0, or 3, or any value other than 4 is put for x; the equation is true on condition that x be 4, and on no other.

Similarly $x^2 - 5x + 6 = 0$ is true when $x = 2$ or when $x = 3$. In the first case $x^2 - 5x + 6 = 0$ becomes $2^2 - 5 \cdot 2 + 6 = 0$, or $4 - 10 + 6 = 0$, or $0 = 0$. In the second case we obtain $3^2 - 5 \cdot 3 + 6 = 0$, or $9 - 15 + 6 = 0$, or $0 = 0$. Plainly the statement obtained is false when -2 is put for x, for then it becomes $(-2)^2 - 5(-2) + 6 = 0$, or $4 + 10 + 6 = 0$, or $20 = 0$. Similarly any value other than 2 or 3, when put for x, gives a relation between numbers which is not true.

Instead of the equality sign, the sign $\equiv$, read *is identical with*, or *is identically equal to*, is often used for emphasis if the equation is an identity.

Thus $3a = 2a + a$ may be written $3a \equiv 2a + a$.

A number or literal expression which, being substituted for the unknown letter in an equation, reduces it to an identity, is said to **satisfy** the equation.

Thus it has been shown that 4 satisfies the equation $4x = x + 12$, and both 2 and 3 satisfy the equation $x^2 - 5x + 6 = 0$. Similarly the literal expression $3a$ satisfies the equation $x - 5 = 3a - 5$.

A number or number symbol is called a **root** *of an equation, if, on substituting it in place of the unknown, the equation becomes an identity.*

A root of an equation satisfies the equation.

The process of finding the root or the roots of an equation is called solving the equation.

The process of checking the solution is really finding out whether the result obtained is a root of the equation or not.

In solving the equation $5k - 4 = 3k + 18$, in Exercise 17, page 35, the student added 4 to each member and subtracted $3k$ from each member. If we indicate this addition of 4 to each member and this subtraction of $3k$ from each member, the equation becomes

$$5k - 4 + 4 - 3k = 3k + 18 + 4 - 3k.$$

Now in the first member $-4 + 4 = 0$, and in the second member $3k - 3k = 0$. Omitting these, the equation becomes $5k - 3k = 18 + 4$.

Comparing this with the original equation, it is seen that -4 has vanished from the first member of the original equation and $+4$ appears in the second member of the last; and that $3k$ has vanished from the second member of the original equation and $-3k$ appears in the first member of the last.

It thus appears that a term may be omitted from one member of an equation, provided the same term, with its sign changed, is written in the other member. This process is called **transposition.**

Hereafter, instead of the method illustrated on page 34, the student will use transposition, as it is more rapid and convenient. He should, however, always remember that the transposition of a term is really the subtraction of that term from each member of the equation.

Like terms in the same member of an equation should be combined before transposing any term.

If we transpose each term of the equation

$$5k - 4 = 3k + 18,$$

it becomes

$$-3k - 18 = -5k + 4,$$

or, reversing the two members,

$$-5k + 4 = -3k - 18.$$

It thus appears that the signs of all the terms of an equation may be changed without destroying the equality. Such a change may also be looked upon as equivalent to multiplying each term of the equation by -1.

Note. Our word *algebra* is derived from the Arabic word for *transposition.* The process by which one passes from the equation $px - q = x^2$ to the equation $px = x^2 + q$ was known as *al-jabr.* This is the first word in the title of an Arabic book on algebra which was translated into Latin. For some reason only this part of the title remained, and by the early part of the seventeenth century *al-jabr*, or algebra, was the common name given to the whole subject.

EXAMPLE

Solve the equation $8x - 5 + 4x + 12 = 13x - 10 - 3x + 29$.

Solution: $8x - 5 + 4x + 12 = 13x - 10 - 3x + 29.$

Combining like terms, $12x + 7 = 10x + 19.$

Transposing, $12x - 10x = 19 - 7.$

Combining like terms, $2x = 12.$

Dividing by 2, $x = 6.$

Check: $8x - 5 + 4x + 12 = 13x - 10 - 3x + 29.$

Substituting 6 for x, $48 - 5 + 24 + 12 = 78 - 10 - 18 + 29.$

Combining, $79 = 79.$

EXERCISES

Solve the following equations and check results:

1. $8x - 2 = 6x + 6.$

2. $4x - 5 = 2x + 10.$

3. $6y - 5 = 9y + 2.$

4. $7y + 3 = 10 + 8y.$

5. $5n - 3 + 21 = 18 + 4n.$

6. $6 + 4n - 15 = 15 - n.$

7. $5n + 3 - 2n = 7 - 4.$

8. $3k + 9 + 5k + 31 = 0.$

9. $6k + 3 - 2k = 27.$

10. $3x - 6 = 34 + 8x.$

11. $2x - 14 - 5x + 4 = 0.$

12. $x + 12 - 11x = -15x + 22.$

13. $5y + 3 = 17 + 3y + 8.$

14. $3y + 5 + 8y + \frac{1}{2} = 0.$

15. $2 - 4h = 3 - 8h + 8.$

16. $3 - 5h + 2 = 7h + 5.$

17. $3h - 25 + 8h - 20 = 0.$

18. $14x - 6x = 22 + 17x - 11x.$

19. $7x - 13 + 8 = x - 27 - 5x.$

20. $4x - 15 - 11x - 18 + 16x - 17 = 0.$

21. $5y - 6 + 3y + 18 - 2y - 25 + 1 = 0.$

22. $0 = 9x - 3 - 4x + 27 + 16x + 18.$

23. $7n - 5 - 4n + 8 = 3n + 18 - 2n - 3.$

EXERCISES

Represent a number:

1. Greater than n by 5.

2. Greater than n by a.

3. Less than n by 3.

4. Less than n by b.

5. Four times n.

6. Three greater than $a + b$.

7. c greater than $a + b$.

8. Five less than $2a - b$.

9. c less than $2a - b$.

10. b less than two times n.

11. a greater than five times n.

12. Seven less than four times n.

13. Eight greater than three times n.

14. One part of 10 is 6. What is the other part?

15. One part of x is 4. What is the other part?

16. One part of 12 is y. What is the other part?

17. One part of x is a. What is the other part?

18. One part of a is x. What is the other part?

19. One part of $x + y$ is z. What is the other part?

20. The sum of two numbers is 18. If one of them is 7, what is the other?

21. The sum of two numbers is 18. If one of them is n, what is the other?

22. The difference of two numbers is 18. If the greater is 34, what is the other?

23. The difference of two numbers is 18. If the greater is n, what is the other?

24. The sum of two numbers is 30. If the smaller is b, what is the other?

25. The sum of two numbers is a. If one of them is 7, what is the other?

26. The sum of two numbers is a. If one of them is x, what is the other?

27. The difference of two numbers is d. If one of them is 6, what is the other?

28. The difference of two numbers is d. If one of them is n, what is the other?

29. What is the excess of 10 over 4? 10 over x?

30. By how much does 25 exceed 9? 25 exceed y? 16 exceed $a + b$?

31. How much greater is 40 than 27? than 40? than a? a than b?

32. How much smaller is 22 than 36? 14 than a? x than y?

33. By how much does $a + 6$ exceed $a - 6$? $a + 6$ exceed $b - 6$? $4x - 3$ exceed $3x - 4$?

34. A is n years old. What will be his age 4 years hence? x years hence? What was his age 3 years ago?

35. A's age is $2n - 3$ years. What will be his age 10 years from now? a years from now? What was his age 8 years ago? a years ago?

36. A and B each have x dollars. If A gives B four dollars, how much will each then have?

37. A and B each have $x + 50$ dollars. If B gives A y dollars, how much will each then have?

38. If A has $x + 30$ dollars and B has $3x - 4$ dollars, express as equations:

(*a*) A and B together have \$200.
(*b*) A has as many dollars as B.
(*c*) A has \$10 less than B.
(*d*) If A gains \$100 and B loses \$50, they have equal amounts.

39. If A's age is x years, B's $2x + 7$ years, and C's $3x - 8$ years, express:

(*a*) The ages of A, B, and C five years hence.
(*b*) The ages of A, B, and C three years ago.

Express each of the following as equations:

(*c*) The sum of the ages of A and B, four years hence, will be 40.

(*d*) The difference of the ages of A and C, six years ago, was 24.

(*e*) In 10 years A will be as old as B is now.

(*f*) Four years ago C was as old as B will be in 10 years.

(*g*) In x years B will be 40.

(*h*) In 2 years the sum of the ages of A, B, and C will be 100.

Translate the following equations into words:

40. $n - 2 = 8.$ **42.** $3n = 27.$ **44.** $18 - n = n - 4.$

41. $n + 3 = 5.$ **43.** $4n - 2 = 16.$ **45.** $3n - 4 = 2n + 8.$

Translate Exercises 46–57 into English, calling m "a number" and n "a second number":

46. $m + n = 20.$ **52.** $m + a = n.$

47. $m - n = 2.$ **53.** $m - b = n.$

48. $2m = n + 6.$ **54.** $3m = 2n.$

49. $3m - 2n = 8.$ **55.** $m = 2n - 6.$

50. $m + n = a.$ **56.** $4 + 3m = 2n + 4.$

51. $m - n = b.$ **57.** $80 - m = 30 + n.$

23. Solution of problems. In the solution of problems in simple equations the following steps are necessary:

1. Read the problem carefully.
2. Represent the unknown number by a letter.
3. Express the conditions stated in the problem as an equation involving this letter.
4. Solve the equation.
5. Check by substituting in the problem the value found for the unknown.

This last is of importance, for substitution in the equation would not detect any errors made in translating the words of the problem into the equation.

The preceding directions for the solution of the various problems leading to simple equations are as definite as can be given. The student will obtain much aid from the study of the typical solutions which occur from time to time. Then one or more careful readings of each problem, a little fixing of the attention upon it, and an application of common sense will insure progress.

In the solution of problems the writing of the equation is nothing more than translating from ordinary speech into the language of algebra. Sometimes it is possible to translate the statement of the problem, word by word, into algebraic symbols.

For example,

Four times a certain number, diminished by 6, gives the same

$4 \quad \times \quad n \quad - \quad 6 \quad =$

result as the number increased by 30.

$n \quad + \quad 30.$

Again,

Seven times A's age two years ago equals five times his age ten years

$7 \quad \times \quad (a-2) \quad = \quad 5 \quad \times \quad (a+10).$

hence.

PROBLEMS

Solve and check the following:

1. To what number must 22 be added so that the sum may be 50?

2. From what number must 15 be subtracted so that the remainder may be 47?

3. What number increased by 9 equals 28?

4. What number diminished by 17 equals 35?

5. What number if doubled and the result diminished by 27 gives 49 as a remainder?

6. What number if trebled and the result diminished by 36 gives twice the original number?

7. Three times a certain number, less 17, equals twice the number, less 1. Find the number.

8. Five times a certain number, increased by 6, equals twice the number, increased by 15. Find the number.

9. Four times a certain number, plus 9, equals seven times the number, minus 33. Find the number.

10. A certain number added to 9 gives the same result as that obtained when the number is subtracted from 71. Find the number.

11. If 6 is added to twice a number, and 10 be subtracted from four times the number, the results are the same. Find the number.

12. The sum of two numbers is 67, and their difference is 5. Find the numbers.

Solution: There are *two* unknowns in this problem, but both can be represented in terms of the same letter, thus:

Let $n =$ the smaller number.

Then $n + 5 =$ the greater number, since the smaller is 5 less than the greater.

The sum of the two numbers is 67.

Therefore $n + n + 5 = 67.$

Combining, $2n + 5 = 67.$

Transposing, $2n = 67 - 5.$

Combining, $2n = 62.$

Dividing by 2, $n = 31$, the smaller number,

and $n + 5 = 36$, the greater number.

Check: $31 + 36 = 67$; $36 - 31 = 5$.

13. The sum of two numbers is 74, and their difference is 12. Find the numbers.

14. The sum of two numbers is 45; the second is 3 less than the first. Find the numbers.

15. The sum of two numbers is 44, and one exceeds the other by 8. Find the numbers.

16. The sum of three numbers is 83. The second is 4 less than the first, and the third is 9 greater than the first. Find the numbers.

17. The sum of three numbers is 66. The second is 3 less than the first, and the third is 18 greater than the second. Find the numbers.

18. The sum of two consecutive numbers is 37. Find the numbers.

19. Find three consecutive numbers whose sum is 39.

20. Find four consecutive numbers whose sum is 90.

21. Find two consecutive even numbers whose sum is 30.

22. Find three consecutive odd numbers whose sum is 87.

23. Find four consecutive even numbers whose sum is 100.

24. A rectangle whose perimeter is 38 feet is 3 feet longer than it is wide. Find its dimensions.

25. A rectangle whose perimeter is 128 feet is 16 feet longer than it is wide. Find the dimensions.

26. The length of a rectangle is 7 feet more than twice the width. Its perimeter is 104 feet. Find the dimensions.

27. The length of a rectangle is 5 feet more than four times its width. Its perimeter is 90 feet. Find the dimensions.

28. At Pittsburg on June 21 the day is 6 hours and 6 minutes longer than the night. How long is the night? the day?

29. A's age is twice B's, and C is 7 years older than A. The sum of their ages is 67 years. Find the age of each.

30. A's age is three times B's, and C is 10 years older than B. Five years hence the sum of their ages will be 60 years. Find the age of each now.

31. A is 10 years older than B, and C is 6 years younger than B. Four years ago the sum of their ages was 46 years. Find the age of each now.

32. A's age is 2 years more than twice B's age, and C's age is 7 years less than A's. Six years hence the sum of their ages will be 70 years. Find the age of each now.

33. In 1907 the yield of corn in the United States exceeded the yield of oats by 1838 million bushels, and the yield of wheat was 120 million bushels less than the yield of oats.

The total yield was 3981 million bushels. Find the number of bushels of each.

34. The north frigid zone and the south frigid zone have the same width, as have also the north temperate zone and the south temperate zone. The torrid zone is 47 degrees wide, or twice the width of the north frigid zone. Together the width of the five zones is 180 degrees. Find the width of each.

35. If the whole number of people in the United States is taken as 100%, 12% more people are engaged in agriculture than in the industries, 8% more in the industries than in commerce, and the rest, 24%, in other pursuits. Find the per cent of people engaged in agriculture, industries, and commerce.

36. In a certain year Montana produced 110 million pounds of copper more than Michigan, and 139 million pounds more than Arizona. If the total production of the three was 514 million pounds, find the amount each produced.

37. The height of the Eiffel Tower, Paris, is 120 feet less than twice the height of the Washington Monument. The latter is 105 feet higher than the Great Pyramid in Egypt, and 107 feet higher than St. Peter's in Rome. If the sum of their heights is 2443 feet, find the height of each.

38. The area of the coal fields of China and Japan is 6000 square miles greater than the area of the coal fields of the United States. The area of the latter exceeds twice that of all other countries (except China and Japan) by 38,400 square miles. If the total area of the coal fields of the world is 471,800 square miles, find the area of the coal fields of China and Japan, of the United States, and of the other countries.

39. In a certain year the production of copper in the United States was 5573 tons less than five times that of Spain and Portugal. These two countries produced 808 tons less than twice the output of Japan. The other countries of the world produced 4615 tons less than five times the output of Japan.

If the world produced 486,084 tons, find the output of the United States, Spain and Portugal, Japan, and the other countries.

40. The area of Asia is 982,000 square miles more than twice that of North America. The area of North America exceeds that of South America by 1,186,000 square miles, and that of Europe by 4,383,000 square miles. The total area of the four continents is 35,692,000 square miles. Find the area of each.

41. The number of United States troops engaged in the Civil War was 15,621 less than nine times the number engaged in the War of the Revolution, which was 266,841 less than the number engaged in the War of 1812. If the total number of United States troops engaged in the three wars was 3,658,811, find the number engaged in each.

42. St. Peter's Cathedral (Rome) has a capacity 29,000 greater than that of St. Paul's Cathedral (London), and 17,000 and 22,000 greater respectively than the Cathedral at Milan and St. Paul's church (Rome). The combined capacity of all is 148,000. Find the capacity of each.

CHAPTER VII

PARENTHESES

24. Removal of parentheses. In solving exercises and problems it is often necessary to inclose several terms in a parenthesis. Sometimes it is necessary to inclose this parenthesis with other terms in a second parenthesis, or even in a third. To avoid confusing the different parentheses, *brackets*, [ ], and *braces*, { }, are also used.

The parenthesis, the brackets, and the braces are called *signs of aggregation.* For convenience, brackets and braces are often spoken of as parentheses.

In the solution of equations and in other exercises it is frequently necessary to remove all signs of aggregation; this removal requires some special study.

The value of $12 + (5 - 3)$ is the same as that of $12 + 5 - 3$, or 14. Similarly $a + (b - c) = a + b - c$.

The plus signs preceding the parentheses in $12 + (5 - 3)$ and $a + (b - c)$ belong to the parentheses and vanish with them, whereas the plus signs *understood* before 5 and b within the parentheses are supplied when we write $12 + 5 - 3$ and $a + b - c$. In the expression $12 + (-5 - 3)$ the sign of 5 must be retained, and we have $12 + (-5 - 3) = 12 - 5 - 3 = 4$.

Therefore we have the

Principle. *A parenthesis preceded by a plus sign may be removed from an expression without changing the signs of the terms which were inclosed by the parenthesis.*

In the expression $12 - (5 - 3)$ the binomial $(5 - 3)$ is to be subtracted from 12. Hence we change the sign of the subtrahend and find the sum of the resulting terms.

Thus $12 - (5 - 3) = 12 - 5 + 3 = 10$. This is obviously correct, for $12 - (5 - 3) = 12 - 2 = 10$.

Similarly $a - (b - c)$ becomes $a - b + c$ when the signs of the subtrahend, $(b - c)$, are changed.

The minus signs preceding the parentheses in $12 - (5 - 3)$ and $a - (b - c)$ vanish with the parentheses, and the plus signs understood before 5 and b within the parentheses are changed when we write $12 - 5 + 3$ and $a - b + c$.

Therefore we have the

PRINCIPLE. *A parenthesis preceded by a minus sign may be removed from an expression, provided the sign of each term which was inclosed by the parenthesis be changed.*

These principles may also be applied to remove the parentheses used to inclose the numbers in Chapter II.

When one parenthesis incloses another, either the outer or the inner parenthesis may be removed first. It is best for the beginner to use the

RULE. *Rewrite the expression, omitting the innermost parenthesis and changing the signs of the terms which it inclosed if the sign preceding it be minus, leaving them unchanged if it be plus.*

Combine like terms that may occur within the new innermost parenthesis.

Repeat this process until all the parentheses are removed.

EXAMPLES

Remove the parentheses from:

1. $8 - (3 - 2a) + (4 - 5a)$.

Solution:
$$\begin{aligned} 8 - (3 - 2a) + (4 - 5a) &= 8 - 3 + 2a + 4 - 5a \\ &= 9 - 3a. \end{aligned}$$

2. $5a - [2a + (-3a - 4b) - (a - 8b) + 4a]$.

Solution:
$$\begin{aligned} 5a - [2a + (-3a - 4b) - (a - 8b) + 4a] \\ &= 5a - [2a - 3a - 4b - a + 8b + 4a] \\ &= 5a - [2a + 4b] \\ &= 5a - 2a - 4b \\ &= 3a - 4b. \end{aligned}$$

EXERCISES

Remove the parentheses and combine like terms:

1. $14-(6-3)-5$.
2. $10+(7-4)-(9-7)$.
3. $(7-3+2)-(6-4)+11$.
4. $11a-(4a-9a)+(6a-a)$.
5. $(2a-5a)-(4a-a-7a)$.
6. $a-(b-c)+(2b-3c)$.
7. $a-b-(c-d)+(a+b)-(b-c)$.
8. $(x-y)-(2y-3x)+(x-4y)$.
9. $x-(x-y+2z)-(3z-y+4)+(x-6)$.
10. $7-[8-(3-10)]-(13-25)$.
11. $a+[2a-(3a-2b)]+(3b-2a)$.
12. $(5x-6y)-[-2x-(4z-y)-2z]$.
13. $[3x-(2y-z)]-[-(3y-2x)-5x]$.
14. $[(a+3)-(x-5)]-[a+3+(x-5)]$.
15. $7-[-6-\{-4+(6-10)\}+11]$.
16. $-5x+[+10x-\{+11x-(2x-7x+4)-3x\}-22]$.
17. $\{4a-[2a-(3a-2b)+4a]-(4b-6)\}$.
18. $2x-3y-[\{+3z-7x-(y+4z)-9x\}+z]$.
19. $(4y-7x)-\{3x-[4x+(7y-4x)-(3y-3x)]\}$.

Sometimes it is necessary to remove some of the signs of aggregation in an expression, leaving others. In the following remove the parentheses, leaving the brackets, and simplify the results as much as possible:

20. $[(a+b)+c]$, $[(a+b)-c]$.
21. $[4x+(3z-5y)]$, $[4x-(3z-5y)]$.
22. $[(a-2b)+(3c-d)]$, $[(a-2b)-(3c-d)]$.
23. $[(4x-3)+(5y-7)]$, $[(4x-3)-(5y-7)]$.
24. $[(x^2-a^2)+(y^2-2a^2)]$, $[(x^2-a^2)-(y^2-2a^2)]$.

25. Inclosing terms in parenthesis. Obviously

$$16 + 9 - 5 = 16 + (9 - 5) = 16 + 4 = 20.$$

Similarly $$a + b - c = a + (b - c).$$

From this we have the

PRINCIPLE. *Two or more terms may be inclosed in a parenthesis preceded by a plus sign, without changing the sign of any of the terms.*

The expression

$$17 + 8 - 3 = 17 - (-8 + 3) = 17 - (-5) = 17 + 5 = 22.$$

Similarly $$a + b - c = a - (-b + c).$$

From this we have the

PRINCIPLE. *Two or more terms may be inclosed in a parenthesis preceded by a minus sign, provided the sign of each term thus inclosed be changed.*

EXERCISES

In the following inclose in a parenthesis preceded by a plus sign all the terms containing the letters x or y, and inclose in a parenthesis preceded by a minus sign all the terms containing the letters a or b:

1. $x^2 - a^2 - 2ab - b^2$. **3.** $y^2 - 4b^2 + 4ab - a^2$.
2. $12ab + x^2 - 9b^2 - 4a^2$. **4.** $10ab + x^2 - a^2 - 25b^2$.
5. $x^2 - b^2 - 4a^2 + 4y^2 - 4ab - 4xy$.
6. $4ab + x^2 - 4b^2 + y^2 - a^2 - 2xy$.
7. $16x^2 - a^2 - 16xy - b^2 + 2ab + 4y^2$.
8. $x^2 - b^2 - 10xy + 12ab - 36a^2 + 25y^2$.
9. $(a^2 - x^2) - (y^2 - b^2)$. **11.** $(3a - 8x + y) - (4b - 7y)$.
10. $(x - a) - (b - y)$. **12.** $(b - 2x) - (a - 2y - 3x)$.
13. $(x^2 - 4ab - b^2) - (a^2 + 3xy + y^2)$.
14. $(2x - 3y - 7a) - (x - 2y - 7b)$.
15. $-(3x^2 + 2a^2 - b^2) - (a^2 - b^2 - y^2 + x^2)$.

CHAPTER VIII

MULTIPLICATION

26. Product of terms containing unlike letters. We assume that the factors of a product may be written in any order. This principle is called the *Commutative Law of Multiplication.*

That is, $2 \cdot 4 = 4 \cdot 2.$

Similarly $a \times b = b \times a.$

As $3 \times b$ is written $3\,b$, $a \times b$ is written ab, $x^2 \times y^2$ is written x^2y^2, and $a \times b \times c$ is written abc.

Further $2\,a^2 \times 3 = 2 \times 3 \times a^2 = 6\,a^2,$

and $2\,a \times 3\,b = 2 \times 3 \times a \times b = 6\,ab.$

Similarly $6\,x^2 \cdot 5\,y^3 = 6 \cdot 5 \cdot x^2 \cdot y^3 = 30\,x^2y^3.$

Also $4\,ab \cdot 3\,z^2 = 4 \cdot 3 \cdot ab \cdot z^2 = 12\,abz^2.$

We have also assumed that the various operations of multiplication in any product may be performed in any order. This principle is called the *Associative Law of Multiplication.*

That is, $(3 \cdot 2)4 = 3(2 \cdot 4)$. Similarly $a(b \cdot c) = (a \cdot b)c$. This merely tells us that a multiplied by the product of b and c is the same as the product of a and b multiplied by c.

Biographical note. Sir William Rowan Hamilton. It is strange that of all the topics treated in this book, the last to be thoroughly understood by mathematicians are those appearing in the first chapters. But in all the sciences it is often most difficult to answer the questions that at first sight seem quite obvious. Any child can ask what electricity is, but the wisest scientist cannot tell. He can only explain what electricity does. It is easy to ask how the earth came to be revolving around the sun with the moon revolving around it, but even the deepest students of astronomy differ in their theories of how it came to be. And so in mathematics, long after many of the more complicated processes of algebra were completely understood, the simple laws of operation of numbers were surrounded with haze. One of the men who did most to clarify the nature of these laws was Sir William Rowan Hamilton (1805–1865). He was born in Dublin, Ireland, where he lived most of his life. He was a precocious boy, and at the age of twelve was familiar with thirteen languages. He devised kinds of numbers that do not follow the same laws as those that we use in algebra, and so threw a flood of light on the nature and properties of these common numbers. He was the first to recognize the importance of the Associative Law, and called it by that name. Most of his works are very advanced in character and are difficult to read.

27. Product of terms containing like letters. By the definition of an exponent (§ 6), $a^2 = a \cdot a$, and $a^3 = a \cdot a \cdot a$.

Therefore $$a^2 \times a^3 = a \cdot a \times a \cdot a \cdot a = a^5 = a^{2+3}.$$

Similarly $$b \times b^3 \times b^5 = b \times b \cdot b \cdot b \times b \cdot b \cdot b \cdot b \cdot b = b^9 = b^{1+3+5}.$$

In like manner $$3^2 \times 3^4 \times 3^5 = 3 \cdot 3 \times 3 \cdot 3 \cdot 3 \cdot 3 \times 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^{11} = 3^{2+4+5}.$$

Also $$ay^2 \times y^3 = ay^5 = ay^{2+3},$$

and $$2\,ab \times 3\,a^2 = 6\,a^3b = 6\,a^{1+2}b,$$

and $$4\,x^2yz \times 5\,xy^3 = 20\,x^3y^4z = 20\,x^{2+1}y^{1+3}z.$$

Therefore we have the

Principle. *The exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.*

This is expressed in general terms, thus:

$$n^a \times n^b = n^{a+b}.$$

The law of signs for the multiplication of positive and negative numbers, given in § 15, applies to literal terms as well.

Thus
$$+2\,a^2 \times (+3\,a^5) = +6\,a^7.$$
$$+2\,a^2 \times (-3\,a^5) = -6\,a^7.$$
$$-2\,a^2 \times (+3\,a^5) = -6\,a^7.$$
$$-2\,a^2 \times (-3\,a^5) = +6\,a^7.$$

For the multiplication of two monomials we have the

Rule. *Keeping in mind the rule of signs for multiplication, write the product of the numerical coefficients followed by all the letters that occur in the multiplier and the multiplicand, each letter having as its exponent the sum of the exponents of that letter in the multiplier and the multiplicand.*

SIR WILLIAM ROWAN HAMILTON

ORAL EXERCISES

Perform the following indicated multiplications:

1. $(3)(-8)$.
2. $(-2)(5)$.
3. $(-7)(-3)$.
4. $(-4x)(3)$.
5. $(-4x)^2$.
6. $(7)(-5a)$.
7. $(3a)(-6)$.
8. $(-2y)^2$.
9. $(-9a)(-10)$.
10. $(-3ax)^2$.
11. $(4a)(-2a)$.
12. $(6abc)^2$.
13. $(-11x)(2x)$.
14. $(7x)(-3x)$.
15. $(-2a)^3$.
16. $(-2a)(-3a^2)$.
17. $(5a^4)(7a^3)$.
18. $(-4x)^3$.
19. $(a^8)(-20a)$.
20. $(-4a^5)(-6a^2)$.
21. $(+6y)^3$.
22. $(4x)(5y)$.
23. $(-3a^2x)^2$.
24. $(3x^2)(-y)$.
25. $(5x^2y)(-2x^3)$.
26. $(-6x^3y^2)^2$.
27. $(-x^4y)(-x^2y^4)$.
28. $(2ax^2)^3$.
29. $(5a^3)(-4a^2)(-3a)$.
30. $(3ax)(-4a^2x)(-2ax^3)$.

28. Multiplication of a polynomial by a monomial. Clearly $2(5+3)$ is equivalent to $2 \cdot 5 + 2 \cdot 3$, each being equal to 16.

Similarly $a(b+c) = ab + ac$. This principle is called the *Distributive Law of Multiplication.*

Therefore, for the multiplication of a polynomial by a monomial, we have the

Rule. *Multiply each term of the polynomial by the monomial and write in succession the resulting terms with their proper signs.*

Example:

$$\begin{array}{ll} & 3x^2 - 2xy + 4y - 5a - 6 \\ & \underline{2xy \qquad\qquad\qquad\qquad\qquad\qquad\quad} \\ \textit{Product,} & 6x^3y - 4x^2y^2 + 8xy^2 - 10axy - 12xy \end{array}$$

Note. It should be kept in mind that the laws of operation that have been mentioned in this chapter, though evident from arithmetic only when the letters represent positive integers, are also valid when the letters stand for negative numbers, fractions, algebraic expressions, or other kinds of numbers that we shall introduce later. The principle which states that the operations on all numbers follow the rules expressed by the commutative, associative, and distributive laws is often called the *Law of Permanence.*

EXERCISES

Multiply:

1. $x + 3$ by $2x$.
2. $7x^2 - 5$ by $3x^3$.
3. $5x^2 - 2x$ by $-4x^2$.
4. $7xy - z$ by $3xy$.
5. $-4x^2 + 5x - 6$ by $6x^3$.
6. $x^3 - 3x^2 + 4$ by $-5x^4$.
7. $x^2 - 2xy + y^2$ by $-3xy$.
8. $a^4 - a^2b^2 + b^4$ by $-a^2b^2$.
9. $-a^2x^2 + 2ax - 7b^2$ by $-4abx$.
10. $7x^3 - 8x^2 - 12x + 6$ by $-\frac{3}{4}x^3$.
11. $-9a^2 - 12ax + 42x^2$ by $\frac{7}{3}ax^3$.

Perform the multiplication indicated:

12. $4(2x - 3)$.
13. $2x(x - y)$.
14. $-8(3x - 7)$.
15. $-9(-4a + b)$.
16. $-3x(2x - 7)$.
17. $-3(x^2 - 2x - 6)$.
18. $5xy(x^2 - 6x + 9)$.
19. $-3x(ax - bx + 3cx^2)$.
20. $-7ab(ax^2 + bx + c)$.

29. Multiplication of polynomials. Clearly $(5 + 3)(7 - 4) = 8 \cdot 3 = 24$. The multiplication may also be performed as follows: $(5 + 3)(7 - 4) = 5(7 - 4) + 3(7 - 4) = 35 - 20 + 21 - 12 = 24$.

Similarly $(2x + 3)(4x - 5) = 2x(4x - 5) + 3(4x - 5) = 8x^2 - 10x + 12x - 15$, or $8x^2 + 2x - 15$.

In general terms $(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$.

This gives for the multiplication of polynomials the

RULE. *Multiply the multiplicand by each term of the multiplier in turn, and add the partial products.*

Example:

$$\begin{array}{lrl}
 & 3x - 2y \qquad\quad & \\
 & \underline{2x + 3y \qquad\quad} & \\
\text{Multiplying by } 2x, & 6x^2 - 4xy \qquad\quad & \text{first partial product.} \\
\text{Multiplying by } 3y, & \underline{+ 9xy - 6y^2} & \text{second partial product.} \\
\textit{Complete product,} & 6x^2 + 5xy - 6y^2 & \text{sum of partial products.}
\end{array}$$

30. Powers. A **power** of a number is the product obtained by using the number as a factor one or more times.

For example, 8, or 2^3, is the third power of 2; 81, or 3^4, is the fourth power of 3, and $32\,x^5$, or $(2\,x)^5$, is the fifth power of $2\,x$.

31. Arrangement. A polynomial is said to be **arranged** according to the *descending* powers of a certain letter when the exponents of that letter in successive terms decrease from left to right. Thus $2\,x^4 - 5\,x^2 - 6\,x + 8$ is arranged according to the descending powers of x. Again, $4 - 2\,y + y^2$ and $x^3 - 3\,x^2y + 3\,xy^2 - y^3$ are arranged according to the *ascending* powers of y.

Whenever it is possible to arrange the multiplier and the multiplicand in a similar order it should be done, as the addition of the partial products is then much more easily performed.

32. Check of multiplication. The work of multiplication can be checked by giving a convenient numerical value to each letter involved and finding the corresponding numerical values of the multiplier, the multiplicand, and the product. The product of the numerical values of the multiplier and the multiplicand should equal the numerical value of the product.

The number 1 is more convenient than any other number to use in checking, but it will not check exponents, since $x^3 = x^5 = x^{10}$, etc., if $x = 1$. It checks merely the coefficients.

If a check on both coefficients and exponents is wanted, the number 2 is the most convenient.

EXAMPLES

1. Multiply $3\,x^3 - 5 + x^2 - 2\,x$ by $6 + x^2 - 5\,x$.

Solution: Arranging both multiplier and multiplicand in descending powers of x and multiplying, we obtain: **Check:** $x = 1$.

$$\begin{array}{lrrrrrrr}
 & \multicolumn{6}{l}{3\,x^3 + x^2 - 2\,x - 5} & = -3\\
 & \multicolumn{6}{l}{\;\;x^2 - 5\,x + 6} & = +2\\ \cline{2-7} \cline{8-8}
 & 3\,x^5 + & x^4 - & 2\,x^3 - & 5\,x^2 & & & -6\\
 & & -15\,x^4 - & 5\,x^3 + & 10\,x^2 + & 25\,x & & \\
 & & & +18\,x^3 + & 6\,x^2 - & 12\,x - & 30 & \\ \cline{2-7}
\textit{Product,} & 3\,x^5 - & 14\,x^4 + & 11\,x^3 + & 11\,x^2 + & 13\,x - & 30 & = -6
\end{array}$$

2. Multiply $10x^3y^2 - 2y^5 + 5x^4y - 4x^2y^3$ by $3x^3 - 7y^3 - 6x^2y$.

Solution:

Arranging terms and multiplying, **Check:** $x = y = 1$.

$$\begin{array}{lllllllr}
5x^4y & +10x^3y^2 & -4x^2y^3 & -2y^5 & & & & =\quad 9 \\
3x^3 & -6x^2y & -7y^3 & & & & & =-10 \\
\hline
15x^7y & +30x^6y^2 & -12x^5y^3 & -6x^3y^5 & & & & -90 \\
 & -30x^6y^2 & -60x^5y^3 & & +24x^4y^4 & +12x^2y^6 & & \\
 & & & -70x^3y^5 & -35x^4y^4 & +28x^2y^6 & +14y^8 & \\
\hline
15x^7y & & -72x^5y^3 & -76x^3y^5 & -11x^4y^4 & +40x^2y^6 & +14y^8 & =-90
\end{array}$$

33. Degree. The **degree** of a term with respect to a certain letter is determined by the exponent of that letter in the term.

Thus x, $3xy$, and $4a^2xz$ are of the first degree in x, and $3xy^2$ is of the second degree in y.

The degree of a term with respect to *two or more* letters is determined by the sum of the exponents of those letters in that term.

Thus $5x^3y$ is of the fourth degree in x and y; $4a^2bc^3$ is of the sixth degree in a, b, and c.

34. Homogeneous expressions. Terms are homogeneous if they are of the same degree with respect to the same letter or letters.

Thus $3a^2b^3$, $4ab^4$, and a^4b are homogeneous terms.

A polynomial is homogeneous if its terms are homogeneous.

For example, $x^3y - 3x^2y^2$ and $3a^4 + a^2b^2 + b^4$ are homogeneous polynomials.

An important property of homogeneous expressions is:

The sum, the difference, the product, or the quotient of any two homogeneous expressions is a homogeneous expression.

This property is useful in checking exponents in multiplication.

Thus, if it be required to multiply $x^2 - 2xy + y^2$ by $x^3 - 3x^2y + 3xy^2 - y^3$, we know beforehand that every term of the product will be of the fifth degree.

EXERCISES

Multiply and check results:

1. $x + 4$ by $x + 3$.
2. $2x + 3$ by $x + 3$.
3. $4x + 7$ by $3x + 2$.
4. $3x - 5$ by $3x + 8$.
5. $3x - 2$ by $2x + 3$.
6. $6 - 4a$ by $5a - 7$.
7. $2x + y$ by $x + 3y$.
8. $2x - 3y$ by $3x - 2y$.
9. $3x - \frac{1}{2}$ by $2x - \frac{1}{3}$.
10. $-3x + 11a$ by $5x - a$.
11. $ax - bx$ by $cx + dx$.
12. $-cx + d$ by $bx - cx^2$.
13. $4x - \frac{1}{3}$ by $6x + \frac{2}{5}$.
14. $x^2 - 5x + 6$ by $x - 3$.
15. $3x^2 - 3x - 7$ by $2x + 4$.
16. $x^2 - xy + y^2$ by $x + y$.
17. $a^2x^2 - 2a^2x + 4a^2$ by $ax + 2a$.
18. $3x^3 - x^2 - 5x$ by $2x^3 - 5x^2$.
19. $2x^2 - 7x + 12$ by $x^2 - 3x - 5$.
20. $a^2 - \frac{1}{2}a + \frac{1}{4}$ by $a^2 - a + \frac{1}{3}$.
21. $x^2 - xy + y^2$ by $x^2 + xy + y^2$.
22. $3x^3 + 5x^2 - x + 2$ by $x^2 - 2x + 1$.

Expand:

23. $(x^3 - x - 5)(2x^2 - 3x - 4)$.
24. $(3x - x^3 + x^2 - 6)(5 - x^2 - 3x)$.
25. $(4a - 5a^2 + 7 + a^3)(6 + a^3 - a + a^2)$.
26. $(5x - 4 + 8x^3)(8 - 5x^2 + 2x^3 - 9x)$.
27. $(x^2y - y^2x)(4xy - 5x^2y)(3x^2y - 7xy^2)$.
28. $(x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z)$.
29. $(a + b + c)^2$.
30. $(c + d - \frac{1}{2})^2$.
31. $(a - 2b + 3c - 4d)^2$.
32. $(x + y + z)^3$.
33. $(x + y)^3 + (x - y)^3$.
34. $(2a - b + 3)^2$.
35. $(2x - 3ay)^3$.
36. $(x + 2y)^2 - (x - 2y)^2$.
37. $(4x - 3y)^2 - (3x + 4y)^2$.
38. $(x - 3)^3 - (2x - 1)^2$.
39. $(x^a - 3)(x^a + 4)$.
40. $(x^{2a} + 5)^2$.
41. $(2x^a - 3)^3$.
42. $(2x^{2a} - 3x)^2$.

CHAPTER IX

PARENTHESES IN EQUATIONS

35. Simple equations involving parentheses. The removal of parentheses is really an easy matter which is governed by simple rules. In handling parentheses, however, it is very easy to acquire careless habits, which are difficult to overcome. Accuracy in such work can be attained only by especial care in removing each parenthesis that is preceded by a minus sign.

EXAMPLES

1. Solve the equation $5(2x - 1) - 3(4x - 6) = 7.$

Solution: Multiplying by the coefficients 5 and 3, this becomes

$$(10x - 5) - (12x - 18) = 7.$$

Removing parentheses,

$$10x - 5 - 12x + 18 = 7.$$

Combining, $-2x + 13 = 7.$

Transposing, $-2x = 7 - 13 = -6.$

Dividing by -2, $x = 3.$

Check: $5(2 \cdot 3 - 1) - 3(4 \cdot 3 - 6) = 7.$

Simplifying, $25 - 18 = 7,$

or $7 = 7.$

Sometimes the square of the unknown number appears and then vanishes, as in the following.

2. Solve the equation $4 + (n - 3)(n - 5) = 15 - (7 - n)(2 + n).$

Solution: Expanding,

$$4 + (n^2 - 8n + 15) = 15 - (14 + 5n - n^2).$$

Removing parentheses,

$$4 + n^2 - 8n + 15 = 15 - 14 - 5n + n^2.$$

Subtracting n^2 from each member and combining,

$$19 - 8n = 1 - 5n.$$

Transposing and combining,

$$-3n = -18.$$

Dividing by -3, $$n = 6.$$

Check: $$4 + (6 - 3)(6 - 5) = 15 - (7 - 6)(2 + 6).$$

Simplifying, $$4 + 3 = 15 - 8,$$

or $$7 = 7.$$

EXERCISES

Solve and check:

1. $5(x - 1) = 30.$

2. $3 + 2(x - 3) = 1.$

3. $7(3x - 2) + 11 = 60.$

4. $4(2x - 5) + 15 = 3(x + 10).$

5. $12y - 2(4y - 7) - 16 = 0.$

6. $9y - 3(2y - 4) = 2(5 - 4y) + 2.$

7. $4 - 2(4y - 3) = 3(y - 5).$

8. $7(y - 3) - 2(4 + y) = 9.$

9. $5(n - 7) + 24 + 4n = 0.$

10. $5n - 9(2n + 4) = 2(n - 9).$

11. $7n - 12 - 2(n - 5) = n - 19.$

12. $4(2n - 7) - 3(4n - 8) + 4 = 2n - 3.$

13. $3h - 2(4h + 8) = 3h - 24.$

14. $5(3h + 1) - 7h = 3(h - 7) + 4.$

15. $(h - 2)(h - 5) = (h + 3)(h + 2).$

16. $(h + 4)(h + 3) - (h + 2)(h + 1) - 42 = 0.$

17. $(x + 4)(x + 6) = (x + 18)(x + 13).$

18. $(k - 7)(5 + k) - (k - 5)(k + 7) + 5 = 0.$

19. $(2x - 5)(4x - 7) = 8x^2 + 52.$

20. $(3y + 5)(4y + 7) - (2y + 3)(6y + 11) - 2 = 0.$

21. $(n + 3)(6n + 5) - (2n + 4)(3n - 8) = 38.$

22. $(x + 3)^2 - (x + 5)^2 = -40.$

23. $(x + 2)^2 - (x - 4)^2 + 48 = 0.$

EXERCISES

1. The length of a rectangle is a and its breadth is b. What is its area? its perimeter?

2. The length of a rectangle is $x - 4$ and its width is 3. What is its area? its perimeter?

3. What is the area of a rectangle whose length is $2x - 4$ and whose breadth is $x + 2$? the perimeter?

4. Each of four horses cost \$100. What was the cost of all?

5. Each of n horses cost \$80. What represents the cost of all?

6. Each of a books cost b cents. What represents the cost of all?

7. What is the total cost of x hats at a cents each, and y hats at b cents each?

8. What is the cost of x horses at $b + 10$ dollars each?

9. Represent the total cost of x chairs at $b + 2$ dollars each, and y chairs at a cost of $c - 3$ dollars each.

10. What is 5% of 16? of x?

11. What is 3% of $x + 120$? of $12x - 300a$?

12. A is n years old. What will three times his age 4 years from now be?

13. If two sums of money are x dollars and $1000 - x$ dollars respectively, express the following as equations:
    (*a*) 4% of the first sum equals \$180.
    (*b*) 3% of the first sum equals 5% of the second.
    (*c*) 5% of the first sum is \$20 less than 4% of the second.

14. A picture is 10 inches wide and 12 inches long and has a frame 2 inches wide. What are the outside dimensions of the frame?

15. If the frame in the preceding were x inches wide, what would represent the outside dimensions of the frame? the

area of the picture and frame? the area of the picture? the area of the frame?

36. Problems involving parentheses. The following problems involve two or more unknowns and the use of parentheses. One of the unknowns can always be represented by a single letter and the others by binomials involving this letter and one or more numbers. It may be necessary in some of the problems to inclose each of these binomials in a parenthesis and to think of them and use them as if they represented a single number. When the student can use a binomial in this way as readily as he uses a single letter, like x, he has made considerable progress in the algebraic way of thinking.

PROBLEMS

1. The sum of two numbers is 88. Three times the less equals twice the greater, plus 29. Find the numbers.

Solution: Here are two unknowns, the greater number and the less. Each can be represented in terms of a single letter as follows:

Let n represent the less number.
Then $88 - n$ must represent the greater.
By the conditions of the problem:

Three times the less = twice the greater + 29.
Hence $3n = 2(88 - n) + 29.$
Simplifying, $3n = 176 - 2n + 29.$
Combining, $3n = 205 - 2n.$
Transposing, $3n + 2n = 205.$
Whence $n = 41$, the less number,
and $88 - n = 47$, the greater number.

Check: $41 + 47 = 88$; $3 \cdot 41 = 2 \cdot 47 + 29$, or $123 = 123$.

2. The sum of two numbers is 49. Twice the greater, minus 13, equals five times the less. Find the numbers.

3. The sum of two numbers is 143. Ten times the less added to five times the greater equals 950. Find the numbers.

4. Separate 45 into two parts such that five times the greater plus four times the less may equal 207.

5. The sum of two numbers is 88. Three times the greater equals five times the less, plus 29. Find the numbers.

6. Separate 93 into two parts so that seven times the less, minus 7, equals six times the greater.

7. Separate 48 into two parts so that twice the greater, minus 7, equals three times the less, minus 5.

8. The sum of two numbers is $12\frac{1}{2}$. Seven times one number minus ten times the other equals 45. Find the numbers.

9. Separate 121 into two parts so that four times the one, increased by 8, equals three times the other.

10. Twice a certain number minus five times another number equals 240. The sum of the numbers is 15. Find the numbers.

11. The sum of two numbers is 14. Nine times the one minus eleven times the other equals zero. Find the numbers.

12. The square of a number plus the square of the next consecutive number is 17 greater than twice the square of the smaller number. Find the numbers.

13. The difference of the squares of two consecutive numbers is 75. Find the numbers.

14. The difference of the squares of two consecutive numbers is 23. Find the numbers.

15. The difference of the squares of two consecutive odd numbers is 104. Find the numbers.

16. The difference of the squares of two consecutive odd numbers is 40. Find the numbers.

17. The product of two consecutive even numbers is 56 less than the square of the greater number. Find the numbers.

18. The product of two consecutive odd numbers equals the square of the smaller increased by 46. Find the numbers.

19. A square has the same area as a rectangle whose length is 8 inches greater and whose breadth is 4 inches less than the side of the square. Find the area of each.

Solution: There are three unknowns in this problem, — the side of the square, the length of the rectangle, and the breadth of the rectangle. The three can be represented in terms of the same letter as follows:

Let $s =$ the side of the square in inches.
Then $s + 8 =$ the length of the rectangle in inches,
and $s - 4 =$ the breadth of the rectangle in inches.

Now the area of the square is $s \cdot s$, or s^2 square inches.

Similarly the area of the rectangle is $(s + 8)(s - 4)$, which, expanded, equals $s^2 + 4s - 32$.

By the conditions of the problem the area of the square equals the area of the rectangle.

Therefore $s^2 = s^2 + 4s - 32$.

Subtracting s^2 from each member,

$$0 = 4s - 32.$$

Whence $s = 8$, the side of the square,
and $s + 8 = 16$, the length of the rectangle,
and $s - 4 = 4$, the breadth of the rectangle.

Therefore the area of the square is $8 \cdot 8$, or 64, square inches, and the area of the rectangle is $16 \cdot 4$, or 64, square inches.

The check is obvious.

20. A square field has the same area as a rectangular field whose length is 30 rods greater, and whose breadth is 20 rods less, than the side of the square. How many acres are there in each field?

21. A tennis court, for two players, is 24 feet longer than twice its breadth. The distance around the court is 210 feet. Find the length and the breadth of the court.

22. A tennis court, for 4 players, is 6 feet longer than twice its breadth. The perimeter of the court is 228 feet. Find the dimensions of the court.

23. The breadth of a basket-ball court is 20 feet less than its length. The perimeter of the court is 80 yards. Find the dimensions.

24. The perimeter of a football field is 780 feet. Its length is 50 yards less than three times its breadth. Find the length and the breadth.

25. The value of 15 pieces of money, consisting of nickels and dimes, is 90 cents. Find the number of each.

Solution: There are two unknowns in this problem, the number of nickels and the number of dimes. Since their sum is 15, the two can be represented in terms of one letter, thus:

Let $d =$ the number of dimes.
Then $15 - d =$ the number of nickels,
and $10 \cdot d =$ the value of the dimes in cents.
Also $(15 - d) 5 =$ the value of the nickels in cents.

The value of the nickels and dimes together is represented by $10 d + (15 - d) 5$.

By the conditions of the problem the value of the nickels and dimes together is 90 cents.

Therefore $10 d + (15 - d) 5 = 90$.
Solving, $d = 3$, the number of dimes,
and $15 - d = 12$, the number of nickels.
Check: $3 \cdot 10 + 12 \cdot 5 = 30 + 60 = 90$.

26. The value of 38 coins, consisting of dimes and quarters, is $5.30. Find the number of each.

27. A collection of nickels, dimes, and quarters amounts to $6.05. There are 5 more nickels than dimes, and the number of quarters is equal to the number of nickels and dimes together. Find the number of each.

28. The value of 40 coins, consisting of nickels and dimes, is $2.90. Find the number of each.

29. A is 20 years older than B. In 10 years A will be twice as old as B. Find the age of each now.

30. A is four times as old as B. In 20 years A will be twice as old as B. Find the present age of each.

31. A's age is 8 years more than twice B's age. Sixteen years ago A was four times as old as B. Find the age of each now.

32. A part of $800 is invested at 3% and the remainder at 4%. The yearly income from the two investments is $30. Find each investment.

Solution: Let $x =$ the number of dollars invested at 3%.
Then $800 - x =$ the number of dollars invested at 4%.
Hence $.03\,x =$ the yearly income from the 3% investment,
and $.04\,(800 - x) =$ the yearly income from the 4% investment.

Therefore, by the conditions of the problem,

$$.03\,x + .04\,(800 - x) = 30. \qquad (1)$$

Multiplying each member of (1) by 100, in order to free the equation of decimals, we obtain

$$3\,x + 4\,(800 - x) = 3000. \qquad (2)$$

Solving (2), $x = 200,$
and $800 - x = 600.$

Hence the 3% investment is \$200 and the 4% is \$600.

Check:

$$\begin{array}{r} 200 \\ .03 \\ \hline 6.00 \end{array} \qquad \begin{array}{r} 600 \\ .04 \\ \hline 24.00 \end{array} \qquad \$6 + \$24 = \$30.$$

33. A part of \$1400 is invested at 5% and the remainder at 6%. The total annual income from the two investments is \$76. Find the amount of each investment.

34. A sum of money at 6% interest and a second sum at 8% yield a total annual income of \$53. The first sum exceeds the second by \$125. Find each.

35. A 5% investment yields annually \$15 less than a 6% investment. If the sum of the two investments is \$1240, find each.

CHAPTER X

DIVISION

37. Division of monomials. Division of numerical terms was explained under Positive and Negative Numbers. On page 22 will be found the rule for this division.

Just as $2 \div 3$ is written $\frac{2}{3}$, so $a \div b$ is written as a fraction, $\frac{a}{b}$, and this result can be simplified no farther.

Similarly $$a^2 \div x^2 = \frac{a^2}{x^2},$$

and $$2a \div 3b = \frac{2a}{3b}.$$

But $$12c^2 \div 4b^2 = \frac{3c^2}{b^2}.$$

In like manner, $-12a \div 6b = -\frac{2a}{b}$. Here the quotient is a fraction, and the minus sign indicates that the fraction is negative.

Similarly $$9x \div (-3y) = -\frac{3x}{y},$$

and $$-24a^2y \div (-6z^3) = +\frac{4a^2y}{z^3}.$$

By the definition of an exponent, $a^5 = a \cdot a \cdot a \cdot a \cdot a$ and $a^2 = a \cdot a$.

Then $$a^5 \div a^2 = \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} = a^3, \text{ or } a^{5-2}.$$

Similarly $$2^6 \div 2^3 = \frac{\cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2 \times 2}{\cancel{2} \times \cancel{2} \times \cancel{2}} = 2^3 = 2^{6-3},$$

and $$ax^3 \div x^2 = \frac{a \cdot x \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = ax, \text{ or } ax^{3-2}.$$

In like manner, $6by^5 \div 2y^3 = 3by^2$, or $3b \cdot y^{5-3}$.

These examples illustrate the

PRINCIPLE. *The exponent of any letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

The foregoing principle expressed in general terms is:

$$n^a \div n^b = n^{a-b}.$$

What this equation means when $b = a$ and when b is greater than a will be explained later.

The law of signs in division may be indicated as follows:

$$\begin{aligned} +ab \div (+a) &= +b. \\ +ab \div (-a) &= -b. \\ -ab \div (+a) &= -b. \\ -ab \div (-a) &= +b. \end{aligned}$$

From what precedes we see that $ax^2 \div x^2 = \dfrac{a \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x}} = a.$

Hence a letter which has the same exponent in divisor and dividend should not appear in the quotient.

Therefore for the division of monomials we have the

RULE. *Divide the numerical coefficient of the dividend by the numerical coefficient of the divisor, keeping in mind the rule of signs for division.*

Write after this quotient all the letters of the dividend except those having the same exponent in divisor and dividend, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

If there are any letters in the divisor unlike those in the dividend, write them under the preceding result as a denominator.

ORAL EXERCISES

Perform the indicated division:

1. $-10 \div 2.$
2. $12 \div (-3).$
3. $-16 \div (-4).$
4. $8a^3 \div a^2.$
5. $-4a^6 \div 2a^2.$
6. $6x^2 \div (-3x).$
7. $-18x^7 \div (-6x^4).$
8. $-25ax^3 \div 5ax.$

9. $12ax^3 \div (-3bx^3)$.

10. $-28ay^4 \div (-7cy^3)$.

11. $70x^4y^9 \div (-10x^2y^8)$.

12. $48ax^5 \div (-16bx^4)$.

13. $-36x^6y^6 \div (-6x^2y^3)$.

14. $63c^3d^5 \div (-9bcd^3)$.

15. $64a^5b^7 \div (-16ab^7)$.

16. $-28a^6b^{12} \div (-7a^4b^8)$.

17. $\dfrac{15x^2y^4z^5}{75xy^4z}$.

18. $\dfrac{42x^{24}y^{56}}{-6x^8y^7}$.

19. $\dfrac{x^a}{x^2}$.

20. $\dfrac{-x^{3a}}{x^a}$.

21. $\dfrac{-17a^9b^8c^7}{51a^7b^8c}$.

22. $\dfrac{39x^{13}y^{26}z^{39}}{-13x^{13}y^{13}z^{13}}$.

23. $\dfrac{-11a^2b^{11}c^{13}}{66ac^{12}}$.

24. $\dfrac{-121a^{11}b^{22}c^{33}}{-11a^{11}b^{11}c^{11}}$.

25. $\dfrac{x^{a+b}}{x^b}$.

26. $\dfrac{x^c}{x}$.

27. $\dfrac{3a^3x^{3a}}{a^2x^a}$.

28. $\dfrac{6x^{2a+3}}{-2x^4}$.

38. Division of a polynomial by a monomial. The division of the binomial (18 − 12) by 3 can be performed in two ways:

Thus $(18-12) \div 3 = 6 \div 3 = 2,$

or $(18-12) \div 3 = \frac{18}{3} - \frac{12}{3} = 6 - 4 = 2.$

Similarly $(ax+bx) \div x = \dfrac{ax}{x} + \dfrac{bx}{x} = a + b.$

Therefore, for the division of a polynomial by a monomial we have the

RULE. *Divide each term of the polynomial by the monomial and write the partial quotients in succession.*

EXERCISES

Perform the indicated division:

1. $\dfrac{6x^2-4x}{2x}$.

2. $\dfrac{9x-18x^4}{-3x}$.

3. $\dfrac{4xy-12x^2}{-4x}$.

4. $\dfrac{9ax^3-12x^5}{-3ax^2}$.

5. $\dfrac{25x^2y+30xy^5}{-5xy}$.

6. $\dfrac{16bx^4-36x^2}{4bx^2}$.

7. $\dfrac{14x^3y^4-28x^5y^6}{7x^2y^3}$.

8. $\dfrac{4x^4y-8x^6y^2+12x^8y^4}{4x^4y}$.

9. $\dfrac{a^3cd^2 - a^2c^3}{a^2cd}$.   10. $\dfrac{ax^4 - bx^3 + cx^2}{-x^2}$.

11. $\dfrac{15\,a^2b^2 + 9\,a^4b^2 - 30\,a^6b^2}{-3\,a^2b^2}$.

12. $\dfrac{16\,a^4b^5c^6 - 24\,a^5b^6c^7 - 48\,a^6b^7c^8}{8\,a^3b^2c}$.

13. $\dfrac{85\,xyz - 51\,x^2yz^2 + 102\,x^3yz^3 - 170\,x^5y^5z}{-17\,xyz}$.

14. $\dfrac{4(x-3) + a(x-3)}{x-3}$.

15. $\dfrac{3\,x(3\,x+4) - 4\,y(3\,x+4)}{3\,x+4}$.

16. $\dfrac{5\,a(2\,x^2 - y) - 3\,b(2\,x^2 - y)}{2\,x^2 - y}$.

17. $\dfrac{(a+b)^4 - 3(a+b)^3}{(a+b)^2}$.   18. $\dfrac{21(x-y)^7 - 35(x-y)^5}{-7(x-y)^5}$.

19. $\dfrac{16(3\,x-4)^4 - 24(3\,x-4)^5 - 48(3\,x-4)^7}{-8(3\,x-4)^4}$.

20. $\dfrac{-5(ac^2 - 2\,d)^3 + x(ac^2 - 2\,d)}{5(ac^2 - 2\,d)}$.

21. $\dfrac{4\,x^4 - 8\,x^{3a} - 6\,x^{2a-2}}{2\,x^3}$.

22. $\dfrac{3\,x^a - 2\,x^{a+1} - x^{a+2} + x^2}{x^2}$.

23. $\dfrac{6\,x^{2a-3} - 12\,x^{4a+4} - 18\,x^{3a+5}}{-3\,x^{2a}}$.

39. Division of one polynomial by another. Division is the reverse of multiplication, and the process of dividing one polynomial by another will be best understood by finding the product of two polynomials and then dividing it by one of them; the other, of course, will be the quotient. A close inspection of the steps in the multiplication (A) which follows will make clear the necessity for each step in the division (B).

$$\begin{array}{rrrr}
4x^2 & -5x & +6 & \\
2x & -3 & & \\ \hline
8x^3 & -10x^2 & +12x & \\
 & -12x^2 & +15x & -18 \\ \hline
8x^3 & -22x^2 & +27x & -18
\end{array} \qquad \text{(A)}$$

Now let $8x^3 - 22x^2 + 27x - 18$ be the dividend and $4x^2 - 5x + 6$ the divisor. Then the quotient must be $2x - 3$.

$$\begin{array}{lrrrr|l}
\text{Dividend,} & 8x^3 & -22x^2 & +27x & -18 & 4x^2-5x+6,\ \text{Divisor} \\ \cline{6-6}
(4x^2-5x+6)2x, & 8x^3 & -10x^2 & +12x & & 2x-3,\ \text{Quotient} \\ \cline{2-5}
 & & -12x^2 & +15x & -18 & \qquad\qquad \text{(B)} \\
(4x^2-5x+6)(-3), & & -12x^2 & +15x & -18 & \\ \cline{2-5}
\end{array}$$

The term having the highest power of x in the dividend, $8x^3$, was obtained by multiplying the term having the highest power of x in the multiplicand by $2x$. If the multiplication were not before us, we could obtain the $2x$ by dividing $8x^3$ by $4x^2$; that is, by dividing the term of highest degree in the dividend by the term of highest degree in the divisor. Multiplying the entire divisor by $2x$, we get the first partial product of the multiplication (A). Subtracting, we get $-12x^2 + 15x - 18$. If the multiplication (A) did not tell us that the second term of the quotient was -3, we could obtain it by dividing $-12x^2$ by $4x^2$; that is, by dividing the term of highest degree in the remainder by the term of highest degree in the divisor. Multiplying the entire divisor by -3 and writing the product under the remainder, we get $(4x^2 - 5x + 6)(-3)$, or $-12x^2 + 15x - 18$, for the second partial product of the multiplication (A). As there is no final remainder the division is said to be exact.

The process of dividing one polynomial by another is expressed in the

RULE. *Arrange the dividend and the divisor according to the descending (or ascending) powers of some common letter, called the letter of arrangement.*

Divide the first term of the dividend by the first term of the divisor and write the result for the first term of the quotient.

Multiply the entire divisor by the first term of the quotient, write the result under the dividend, and subtract, being careful *to write the terms of the remainder in the same order as those of the divisor.*

Divide the first term of the remainder by the first term of the divisor for the second term of the quotient and proceed as before until there is no remainder, or until the remainder is of lower degree in the letter of arrangement than the divisor.

If there is no remainder, the result of division may be expressed as follows:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient}.$$

If there is a remainder, the result is expressed as follows:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \frac{\text{Remainder}}{\text{Divisor}}.$$

This last corresponds to what is done in arithmetic in dividing 17 by 5, which is written $\frac{17}{5} = 3\frac{2}{5}$. This means that $\frac{17}{5} = 3 + \frac{2}{5}$, the plus being understood.

Note. We saw on page 1 that it is customary to represent the product of two letters by placing one after the other with no sign between them. Thus ab means a times b. But addition, not multiplication, is implied by placing the fraction $\frac{2}{5}$ after the number 3. This practice comes down to us from the Arabs, who denoted all additions by placing the number symbols in succession without any sign of operation. The later Greeks also had the same notation.

EXAMPLES

1. Divide $38x + 2x^4 - 7x^2 - 24 - 7x^3$ by $6 + x^2 - 5x$.

Solution: Arranging terms and dividing,

$$\begin{array}{r|l}
\text{Dividend, } 2x^4 - 7x^3 - 7x^2 + 38x - 24 & x^2 - 5x + 6, \text{ Divisor} \\
\underline{2x^4 - 10x^3 + 12x^2} \qquad\qquad & \underline{2x^2 + 3x - 4}, \text{ Quotient} \\
3x^3 - 19x^2 + 38x \qquad\quad & \\
\underline{3x^3 - 15x^2 + 18x} \qquad\quad & \\
-4x^2 + 20x - 24 & \\
\underline{-4x^2 + 20x - 24} &
\end{array}$$

Check: Let $x = 1$. Then the dividend $= 2$, the divisor $= 2$, and the quotient $= 1$; and $2 \div 2 = 1$.

The student must avoid checking by any number which makes the divisor zero.

2. Divide $8xy^2 + 8x^3 - 7y^3 - 12x^2y$ by $4x^2 + y^2 - 4xy$.

Solution: Arranging terms and dividing,

$$\begin{array}{rr|l}
\text{Dividend,} & 8x^3 - 12x^2y + 8xy^2 - 7y^3 & 4x^2 - 4xy + y^2, \text{ Divisor} \\
 & \underline{8x^3 - 8x^2y + 2xy^2 \phantom{{}-7y^3}} & \overline{2x - y}, \text{ Partial Quotient} \\
 & -4x^2y + 6xy^2 - 7y^3 & \\
 & \underline{-4x^2y + 4xy^2 - y^3} & \\
 & 2xy^2 - 6y^3, \text{ Remainder} &
\end{array}$$

The total quotient is $2x - y + \dfrac{2xy^2 - 6y^3}{4x^2 - 4xy + y^2}$.

Check: Let $x = y = 1$. Then the dividend $= -3$, the divisor $= 1$, and the quotient $= -3$; and $-3 \div 1 = -3$.

General Check for Division. (*a*) When the division is exact. Multiply the divisor by the quotient. The product should be the dividend.

(*b*) When there is a remainder. Multiply the divisor by partial quotient and add in the remainder. The result should be the dividend.

EXERCISES

Divide:

1. $x^2 + 7x + 12$ by $x + 3$.
2. $x^2 - 2x - 15$ by $x - 5$.
3. $x^2 + 5x + 6$ by $x + 3$.
4. $-7x + 6 + x^2$ by $x - 1$.
5. $6x^2 - 13x + 6$ by $2x - 3$.
6. $25x^4 + 30x^2 - 7$ by $7 + 5x^2$.
7. $12a^2 - 21 + 19a$ by $4a - 3$.
8. $-8 + x^3 + 4x - 2x^2$ by $x - 2$.
9. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
10. $x^3 - 15x^2 + 65x - 63$ by $x - 7$.
11. $5x^2 + 5x - 25x^3 - 1$ by $5x^2 - 1$.
12. $2x^3 - 14x^2 + 14x + 12$ by $2x - 4$.

13. $3a^3 + 28a^2 + 89a - 140$ by $3a - 5$.

14. $37x + 6x^3 - 24 - 23x^2$ by $2x - 3$.

15. $53a + 8 - 53a^2 + 12a^3$ by $4a^2 - 7a - 1$.

16. $15a^3 - 56a^2 + 99a - 70$ by $3a^2 - 7a + 10$.

17. $23a^2 + a^4 - 55a + 11a^3 - 140$ by $a^2 - 5$.

18. $4a^3 + 1 + a^4 + 4a + 6a^2$ by $1 + a^2 + 2a$.

19. $a^4 - 8a^3 + 24a^2 - 32a + 16$ by $a^2 - 4a + 4$.

20. $40x - 31x^2 + 21 + x^4 + 4x^3$ by $x^2 - 3 - 7x$.

21. $11x - 42x^2 + 10x^4 - 27x^3 - 36$ by $9 + 2x^2 - 5x$.

22. $x^4 - 3a^2x^2 + a^4$ by $x^2 - ax + a^2$.

23. $a^4 + 4b^4 + 3a^2b^2$ by $2b^2 + a^2 - ab$.

24. $16x^4 - 60x^2y^2 + 25y^4$ by $4x^2 - 10xy - 5y^2$.

25. $9a^4 + 49b^4 + 26a^2b^2$ by $7b^2 + 3a^2 + 4ab$.

26. $4a^8 - 44a^4b^4 + 100b^8$ by $2a^4 - 10b^4 + 2a^2b^2$.

27. $25x^3 - 10x^2 + 40x - 18$ by $5x - 6$.

28. $x^3 - y^3$ by $x - y$.

29. $a^3 - 125b^3$ by $a - 5b$.

30. $a^6 + 343b^3$ by $a^2 + 7b$.

31. $x^4 - 16$ by $x + 2$.

32. $y^5 - 5y^2 - 3000$ by $y - 5$.

33. $x^4 + y^4$ by $x - y$.

34. $x^4 + y^4$ by $x + y$.

35. $x^4 - y^4$ by $x + y$.

36. $x^4 - y^4$ by $x - y$.

37. $x^5 + y^5$ by $x + y$.

38. $x^5 + y^5$ by $x - y$.

39. $x^5 - y^5$ by $x + y$.

40. $x^5 - y^5$ by $x - y$.

41. $x^{2a} - 5x^a + 6$ by $x^a - 3$.

42. $x^{6a} - 7x^{3a} + 12$ by $x^{3a} - 4$.

43. $x^{2a} + 2x^{a+1} + 3x^a + 3x$ by $x^a + 2x$.

CHAPTER XI

EQUATIONS AND PROBLEMS

40. Equations involving literal coefficients. The most general form of a simple equation in one unknown is one in which the unknown occurs with literal coefficients. The solution of such an equation frequently involves division of polynomials.

EXAMPLE

Find the value of x in $ax + 4a = a^2 + 2x + 4$ and check the result.

Solution: $ax + 4a = a^2 + 2x + 4.$

Transposing, $ax - 2x = a^2 - 4a + 4.$

Writing the coefficients of x as a binomial,

$$(a-2)x = a^2 - 4a + 4.$$

Dividing both members by the coefficient of x,

$$x = \frac{a^2 - 4a + 4}{a-2} = a - 2.$$

Check: Substituting $a - 2$ for x in the original equation, it becomes:

$$a(a-2) + 4a = a^2 + 2(a-2) + 4.$$

Simplifying, $a^2 - 2a + 4a = a^2 + 2a - 4 + 4.$

Combining, $a^2 + 2a = a^2 + 2a.$

EXERCISES

Solve for x and check:

1. $x + 2a = 6a.$
2. $x + a = b.$
3. $bx + b = 4b.$
4. $cx + c^2 = 6c^3.$
5. $5(b - x) = 10b.$
6. $bx - (b + c) = 5b - c.$
7. $3ax - ab = 2ax - ac.$
8. $4bx - 7a^2b = 6ab^2 + 3bx.$

9. $ax + bx = ac + bc.$

10. $a^2x + 1 - a^4 - x = 0.$

11. $ax + 2ab = 2a^2 + bx.$

12. $ax - a^3 - 4 = 3a - x.$

13. $4b^2c^2 + (a + bx)c = (a - bx)c.$

14. $ax - ac + bc = 2ac - 5bc + 2bx.$

15. $(x + a)(x + b) = x^2 + 2a^2 + 3ab.$

16. $15(x - a) - 6(x + a) = 3(5a - 3x).$

17. $4x - cx - 8 + 2a + 6c = 6a - 3ac + 2cx.$

18. $9ab + (x - 3a)(x - 3b) = (x + 3a)(x - 3a) - 9a^2.$

19. $(5a - 4b)x - 5(b^2 + 4a^2 + 6ab) = 10b^2 - 3(2a^2 + 3bx) - a(2x - b).$

20. $a^2x + 3ax + 10a = a^3 + x + 3.$

41. Uniform motion. If a train travels for 8 hours at an average rate of 40 miles an hour, the total distance traversed is 8 × 40, or 320 miles. This illustrates **uniform** motion involving:

1. Time measured in seconds, minutes, hours, etc.

2. Rate (velocity), or the distance traveled in a unit of time, one second, one hour, or one day.

3. Distance (total) measured in standard units of length as feet, or inches, or meters, or kilometers, etc.

Time (t), rate (r), and distance (d) are connected by the relation

$$d = r \times t.$$

On this simple equation a large number of problems in algebra and in physics are based.

Biographical note. Sir Isaac Newton. Sir Isaac Newton (1642–1727) was probably the keenest mathematical thinker who ever lived. He was the son of a farmer of slender means, and as a boy was rather lazy. It is said, however, that his complete victory over a larger boy in a fight at school led him to feel that perhaps he could be equally successful in his studies if he really tried. His ambition and interest being once roused, he never ceased to apply himself during the rest of his long life.

His most important scientific achievement was the discovery and verification of the laws of motion. In his great work called the "Principia"* he showed by mathematical reasoning that all bodies, great and small, — the planet revolving around the sun, as well as the apple falling from the tree, — follow the same laws. His greatest discovery in pure mathematics was that of a method called the calculus, which is the basis of most of the advances in mathematics and in theoretical physics made since his time.

But important as was Newton's mathematical work, his most significant contribution to mankind was an idea, — the idea that the world in which we live is not independent of the rest of the universe, but that every smallest particle of matter is connected with the most remote planet and star; that we cannot think of ourselves as the center of all things, but that we merely occupy our place in a system of universal law.

EXAMPLE

A pedestrian traveling 4 miles per hour is overtaken 14 hours after leaving a certain point by a horseman who left the same starting point 8 hours after the pedestrian. Find the rate of the horseman.

Solution: This is a problem in uniform motion, involving the distance, the rate, and the time of a pedestrian and a horseman respectively. By a careful reading of the problem one discovers that the time for each was a different number of hours, that each went at a different rate, but that each traveled the same distance. Hence the equation will be formed by expressing d in terms of r and t for both the pedestrian and the horseman and then equating the two expressions for d.

By the conditions:

	t, or time in hours	r, or rate in miles per hour	Distance, $d = r \times t$
Pedestrian	14	4	$56 = 4 \times 14$
Horseman	6	x	$6 \cdot x$

Hence $6x = 56,$

and $x = 9\frac{1}{3}.$

Check: $4 \cdot 14 = 56;\ 9\frac{1}{3} \cdot 6 = 56.$

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* A copy of this book, presented to the College by Newton himself, may be seen in the library of Yale University.

SIR ISAAC NEWTON

PROBLEMS

A and B start from the same place at the same time and travel in opposite directions:

1. A goes 8 miles per hour and B 10 miles per hour. In how many hours will they be 120 miles apart? 180 miles apart?

2. A travels twice as fast as B. In 5 hours they are 135 miles apart. Find the rate of each.

3. A travels 2 miles per hour more than B. After 8 hours they are 96 miles apart. Find the rate of each.

4. A goes 4 miles per hour more than B. After 6 hours the distance between them is 168 miles. Find the rate of each.

5. B goes 3 miles per hour less than A, and travels $\frac{3}{4}$ as fast as A. Find the rate of each. After how many hours will the distance between them be 168 miles?

6. A travels 3 hours and stops, and B travels 5 hours. Then they are 77 miles apart. A's rate is twice B's. Find their rates and the distance each has traveled.

7. B travels 9 hours at a rate of 4 miles per hour less than A. A travels an equal distance in 3 hours less time, and then stops. How far are they apart at the end of 9 hours?

A and B start at the same time from two points 144 miles apart and travel toward each other until they meet. Find the rate of each:

8. If they travel at the same rate and meet in 8 hours.

9. If A travels 2 miles per hour less than B and they meet in 9 hours.

10. If B travels three times as fast as A and they meet in 12 hours.

11. If they meet in 6 hours and B travels 24 miles more than A.

12. If they meet in 8 hours and B goes 2 miles per hour more than A.

13. If they meet in 9 hours and A travels 4 miles per hour more than B.

Find the number of hours from the start until the time of meeting:

14. If B goes 6 miles per hour more than A and travels twice as far as A.

15. If A travels 6 miles per hour and B 9 miles per hour, but B is delayed 4 hours on the way.

16. If A is delayed 3 hours and B is delayed 5 hours, and their rates are 7 miles and 9 miles per hour respectively.

17. A and B start from the same place at the same time and travel in opposite directions. A travels 3 miles per hour and B 4 miles per hour. In how many hours will they be 42 miles apart?

18. A and B start at the same time from two points 72 miles apart and travel toward each other. A travels 8 miles per hour and B 10 miles per hour. In how many hours will they meet?

19. The distance from Kansas City to St. Louis is 285 miles. A train running 37 miles per hour leaves Kansas City for St. Louis at the same time a train running 38 miles per hour leaves St. Louis for Kansas City. In how many hours will they meet?

20. A starts from a certain place and travels 8 miles per hour. Four hours later B starts from the same place and travels in the same direction at the rate of 10 miles per hour. How many hours does B travel before overtaking A?

21. Two bicyclists 108 miles apart start at the same time and travel toward each other. One travels 10 miles per hour, the other 12 miles per hour. The latter is delayed 2 hours on the way. In how many hours will they meet, and how far has each traveled?

22. A passenger train starts 2 hours later than a freight train, from the same station but in an opposite direction. The rate of the passenger train is 42 miles per hour and the rate of the freight train is 24 miles per hour. In how many hours after the passenger train starts will the two trains be 246 miles apart?

23. A messenger going at the rate of 8 miles per hour has journeyed 2 hours when it is found necessary to change the message. At what rate must a second messenger then travel to overtake the first in 6 hours?

24. A passenger train and a freight train start together from the same station and move in the same direction on parallel tracks at the rate of 45 miles and 18 miles per hour respectively. How much time will have elapsed before the passenger train will be 144 miles ahead of the freight train?

(Problems 25–32 may be solved without using equations.)

The distance from P to Q is 108 miles. A and B leave P at the same time and travel at different rates toward Q. The one who reaches Q first at once returns. Find the distance each has traveled when they meet:

25. If A's rate is 9 miles per hour and B's is 15.

26. If B travels five times as fast as A.

27. If A travels 56 miles more than B.

28. If they meet 24 miles from Q and B travels faster than A.

Find the rate of each if A travels faster than B:

29. If they meet in 6 hours halfway between P and Q.

30. If they meet in 6 hours $\frac{2}{3}$ of the way from P to Q.

31. If they meet in 6 hours 12 miles from Q.

32. If they meet in 6 hours 96 miles from P.

33. If A travels 4 miles per hour more than B and meets B in 12 hours.

Find the distance each travels:

34. If they meet in 6 hours and A travels 2 miles per hour more than B.

35. If A travels 4 miles per hour more than B and they meet in 9 hours.

The velocity of a bullet continually decreases from the instant it leaves the gun. This is due to the resistance of the air. In the following problems consider the velocity of sound as 1100 feet per second.

36. Two and one-half seconds after a marksman fires his rifle he hears the bullet strike the target which is 550 yards distant. Find the average velocity of the bullet.

37. One and three-fourths seconds after a marksman fires his revolver he hears the bullet strike the target 50 rods distant. Find the average velocity of the bullet.

38. A marksman fires at a target 1000 yards distant. The bullet passes over a boy, who hears the sound of it striking the target and the report of the gun at the same instant. The velocity (average) of the bullet is 1650 feet per second. Find the distance of the boy from the target.

REVIEW EXERCISES AND PROBLEMS

1. Simplify $5a - 7x + 3b - 10c - 14a + 12y - 8x + 12a - 11x + 9b$.

2. Simplify $2cd - a^2b + 7cd^2 - 12ab + 17ad - cd + 4c^2d$.

3. Add $10x - 9 + 4ax - 2cd$, $-6ax - 15x + 12cd$, $10 - ax + 5cd$, and $-6cd + 11x - 19 - 7ax$.

4. Add $a^3 - 3 + 4a^2 - a$, $-a + 3a^2 - 5a^3$, $-3a^2 + 6a^3 - 2a + 8$, $-a - 5a^2 - 2a^3 + 5$, and $2a^3 - 3a^2 + 4a - 5$.

5. Add $Ax^3 + Bx^2 + Cx$ and $Ax^2 + Bx + C$.

6. Add $7(a-b) - 10(c-d) + 8(x-y)$, $-6(c-d) - 4(a-b) - 7(x-y)$, and $12(x-y) - 15(c-d) + 7(a-b)$.

7. Add $-2(x+y) + 3(x-y) - 4(a+b)$, $8(x-y) - 9(x+y) + 4(a-b)$, and $-(a+b) + (x-y) - 10(x+y)$.

8. Add $\frac{1}{3}ax + \frac{2}{3}ax^2 - \frac{5}{6}a^2x^2 + a$, $\frac{1}{6}ax^2 - ax - 9a + 2a^2x^2$, and $a^2x^2 + \frac{1}{2}ax^2 - \frac{1}{4}a$.

9. What must be added to $2x^2 - 3xy + y^2 - 1$ to give $x^2 + 10xy - 9 - 5y^2$?

10. What must be added to $3\,a^2y - 7\,ay^2 + z - 5\,ay$ to give $-a^2y + 12\,ay^2 - ay - 15$?

11. What must be added to $-17\,a^2bcd^2 - 4\,ab^2c^2d^3 + 3\,abcd + ax$ to give $3\,ab^2c^2d^3 - 12$?

12. What must be added to $27xy + 13\,z - 12 + ab$ to give 0?

Remove the parentheses and collect terms:

13. $(a + b) - (a - b) + 4\,a - (a + 3\,b) + c.$

14. $15\,x - 2\,(12\,x - 16\,y) + xy + 3\,(3\,x - 4\,xy) - x.$

15. $ab - (7\,x + 4\,y - 2\,z) + 42 + 5\,(5\,ab - x - y).$

16. $[x - (-4\,x + 8\,y + 2\,z) + b - 4\,(3\,x - z)] - 5\,b.$

17. $[ab - \{-5\,ac + 7\,(d - a) + 4\,ca + 7\,d\} - 6\,ba] + 13\,a.$

18. $\{5\,x^2 - 2\,[-4 - (2\,x^2 - 3\,x)] + 5\} - (x^2 + 4\,x - 2).$

Perform the indicated multiplication:

19. $(x^4 - 4\,x^2y^2 + 4\,y^4)\,(x^4 + 4\,x^2y^2 + 4\,y^4).$

20. $(4\,x^4 - x^2 + 5)\,(3\,x^3 - x + 7).$

21. $(a + b + c)\,(a^2 + b^2 + c^2 - ab - ac - bc).$

22. $(x + y)\,(x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6).$

23. $(5\,x - 9\,x^5 + x^2 - 4\,x^4 - 3\,x^3 + 10)(-x^3 + 3\,x - x^2 + 2\,x^4).$

24. $(3\,x^2 - 4\,y)\,(9\,x^4 + 16\,y^2)\,(3\,x^2 + 4\,y).$

25. $(a - 3\,b + 2\,c)^2.$

26. $(x - y + z)^3.$

27. $(x^a - y^b)\,(x^a + y^b).$

28. $(x^a + 2\,y^b + 3\,z^c)\,(x^a - 3\,y^b).$

Divide:

29. $6\,a^5 - 13\,a^4 + 4\,a^3 + 3\,a^2$ by $2\,a^2 - 3\,a^2 + a.$

30. $-30\,a^4 - 11\,a^3 + 82\,a^2 + 12\,a - 48$ by $3\,a^2 + 2\,a - 4.$

31. $10\,x^2 + 20\,x^3 - 11\,x^5 + 10\,x^6 - 3\,x^4 + 2$ by $-3\,x^2 - 2 + 2\,x + 5\,x^3.$

32. $a^8 - 3\,a^6 + a^4 - 7\,a^2 + 3$ by $(a^2 - 1)^2.$

33. $a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 7\,ab^4 - b^5$ by $(a - b)^2.$

34. $(6\,a^2 + 5\,a - 6)\,(2\,a^2 - 13\,a + 20)$ by $(3\,a - 2)\,(2\,a - 5).$

35. $x^3 - 27\,y^3 + 64 + 36\,xy$ by $x - 3\,y + 4.$

Find the value of y in the following equations:

36. $6y - 20 + 5y - 18 = 36y - 4 - 40y + 9.$

37. $4y - 3(6 - y) + 2 = 6(y - 2) - 13y + 24.$

38. $6y - 4(1 - 2y) + 10 = 4(2y - 5) - 4y + 1.$

39. $(y + 7)(y - 11) = (y + 8)(y - 5) - 2.$

40. $(y - 3)^3 = (y^2 - 6y + 9)(5 + y) - 8y^2.$

41. $y^2 + a^2 - (y + 1)^2 - (a + 1)^2 = 0.$

42. $3ay + 6ac + 4b^2 = 3ab + 4by + 8bc.$

Find the value of x, if $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$:

43. When $a = 1$, $b = +10$, and $c = -11$.

44. When $a = 5$, $b = -6$, and $c = -8$.

Find the value of x, if $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$:

45. When $a = 3$, $b = 5$, and $c = -8$.

46. When $a = 2$, $b = 7$, and $c = -22$.

47. If $f = 12$, $m = 18$, and $v = 20$, find the value of s in the equation $fs = \frac{1}{2}mv^2$.

48. If $a = 32.2$, $m = 7$, $t = 40$, find the value of e in the equation $e = \frac{1}{2}m(at)^2$.

49. If $s = \dfrac{a + b + c}{2}$, find the value of $\sqrt{s(s - a)(s - b)(s - c)}$ when $a = 3$, $b = 4$, and $c = 5$.

50. If the square of a certain number is increased by 38, the result is equal to the product of the next two consecutive numbers. Find the numbers.

51. The square of a certain odd number is 98 less than the product of the next two consecutive odd numbers. Find the numbers.

52. If the square of a certain odd number is increased by 47, the result is equal to the product of the next two consecutive even numbers. Find the numbers.

53. The product of two consecutive numbers is 42 less than the product of the next two consecutive numbers. Find the numbers.

54. The ages of two persons are respectively 42 years and 15 years. In how many years will the elder be twice as old as the younger?

55. The ages of two persons are respectively 36 years and 16 years. How many years ago was the older person three times as old as the younger?

56. The ages of two men are respectively 52 years and 21 years. How many years hence will the older man be twice as old as the younger?

57. The ages of two persons are 87 years and 42 years respectively. How many years ago was the elder four times as old as the younger?

58. A collection of quarters, dimes, and nickels, containing 32 coins, is worth $3.60. There being twice as many nickels as quarters, find the number of each.

59. A bullet is fired from a rifle at a speed which would average 1280 feet per second. Six seconds later the marksman hears it strike the target. The velocity of sound is 1120 feet per second. Find the distance to the target.

60. The leader in some games proposed to tell the age of the others thus: each was to add 12 years to his age, to multiply the sum by 3, to subtract 36 from the product, and then to add his age. Each in turn announced his final result and the leader at once gave the correct age. What did he do to each result to obtain the proper age?

61. The per cent of the population of the United States under 20 years exceeds by 5% the population between 20 years and 60 years. The per cent between 20 years and 60 years is nine times the per cent above 60 years. Find the per cent of the population under 20 years, between 20 years and 60 years, and over 60 years.

62. The length of the St. Gothard tunnel exceeds that of Mount Cenis (Italy) by 9000 feet. The length of the St. Gothard tunnel is 1320 feet less than twice the length of the tunnel at Hoosac, Massachusetts. If the sum of the lengths of these tunnels is 113,760 feet, find the length of each.

63. The height of the first cascade of the Yosemite waterfall exceeds that of Staubbach Falls by 520 feet and is 60 feet more than four times the height of the falls of the Zambezi. The height of the latter is 32 feet more than twice the height of Niagara Falls. The sum of the four heights is 3004 feet. Find each.

64. The combined weight of a cubic foot of mercury, a cubic foot of water, and a cubic foot of alcohol is 962.2 pounds. Alcohol weighs 11.5 pounds per cubic foot less than water, while a cubic foot of mercury weighs 32.7 pounds more than sixteen cubic feet of alcohol. Find the weight of each per cubic foot.

65. If the average annual rainfall at Boston were 3 inches less, it would be one third the rainfall of Neahbay, Washington. The annual rainfall of Boston is 1 inch greater than that of St. Louis and 1 inch less than five times that of San Diego, California. If these places together have a total annual rainfall of 219 inches, find the rainfall at each place.

66. From tables which have been compiled it is found that a person 10 years old may expect to live 5.39 years longer than one 21 years old and 15.31 years longer than one 45 years old; and a person 45 years old may expect to live .94 of a year less than twice as long as one 65 years old. If four people (one at each of these ages) may expect to live a total of 109.42 years, how many years may each expect to live?

CHAPTER XII

IMPORTANT SPECIAL PRODUCTS

42. The square of a binomial. The multiplication

$$\begin{array}{l} a\ +b \\ \underline{a\ +b\qquad\quad} \\ a^2 + \quad ab \\ \underline{\qquad\quad ab + b^2} \\ a^2 + 2\,ab + b^2 \end{array}$$

gives the formula

$$(a + b)^2 = a^2 + 2\,ab + b^2.$$

This may be expressed in words as follows:

I. *The square of the sum of two terms is the sum of the squares of the terms plus twice their product.*

Similarly
$$(a - b)^2 = a^2 - 2\,ab + b^2,$$

which may be expressed in words as follows:

II. *The square of the difference of two terms is the sum of their squares minus twice their product.*

EXERCISES

Square the following either by I or II:

1. 16.

Solution: $16^2 = (10 + 6)^2 = 100 + 36 + 2 \cdot 10 \cdot 6 = 256.$

2. 49.

Solution: $49^2 = (40 + 9)^2$ or $(50 - 1)^2$.
Using the latter, $(50 - 1)^2 = 2500 + 1 - 2 \cdot 50 \cdot 1 = 2401.$

3. 15.

4. 17.

5. 18.

6. 19.

7. $x + y$.

8. $x - c$.

9. $x + 1$.

10. $a + 2$.

11. $b^2 + 3$.

12. $c - 4$.
13. $2\,c^2 + d$.
14. $3\,x - y$.
15. $x + 2\,y$.
16. $a - 4\,b$.
17. $y^5 - 2$.
18. $2\,a + 1$.
19. $3\,a^2 + 2$.
20. $4\,a + 3$.
21. $2\,a^2 - 3\,b$.
22. $3\,c^2 - 4\,a^2$.
23. $2\,x^3 - 6$.
24. $3\,xy + y^2$.
25. $7\,x^2 - 3\,y^4$.
26. $8\,y^3 + 2\,bx$.
27. $6\,xy - 2\,a^2$.
28. $4\,xy - 2\,x^2$.
29. $3\,x^4 - 5\,x^6$.
30. $2\,x^2 - 6\,x^3y$.
31. 41.
32. 92.
33. 101.
34. 202.
35. 1001.

Perform mentally the indicated division:

36. $\frac{a^2 + 2\,ab + b^2}{a + b}$.
37. $\frac{a^2 - 2\,ac + c^2}{a - c}$.
38. $\frac{a^2 - 4\,a + 4}{a - 2}$.
39. $\frac{9\,x^2 - 6\,x + 1}{3\,x - 1}$.
40. $\frac{y^2 + 40\,y + 400}{20 + y}$.
41. $\frac{9\,x^2 - 12\,xy + 4\,y^2}{2\,y - 3\,x}$.
42. $\frac{4\,a^2 - 20\,a + 25}{-5 + 2\,a}$.
43. $\frac{4\,x^2 - 12\,xy + 9\,y^2}{3\,y - 2\,x}$.

Find a binomial divisor for each of the following trinomials:

44. $x^2 - 2\,xy + y^2$.
45. $a^2 + 2\,a + 1$.
46. $a^2 + 4\,a + 4$.
47. $4\,x^2 + 4\,x + 1$.
48. $16\,x^2 - 8\,x + 1$.
49. $16\,x^2 - 8\,xy + y^2$.
50. $9\,x^4 - 6\,x^2 + 1$.
51. $16\,y^6 - 40\,y^3 + 25$.

State the two binomials whose product is:

52. $c^2 + 2\,cd + d^2$.
53. $a^2 - 2\,a + 1$.
54. $x^2 - 10\,x + 25$.
55. $4\,a^2 - 4\,a + 1$.
56. $4\,x^2 + 12\,x + 9$.
57. $9\,a^2b^2 - 6\,ab + 1$.
58. $16\,x^4y^2 - 24\,x^2ya + 9\,a^2$.
59. $25\,a^4b^6 - 60\,a^2b^3c^2 + 36\,c^4$.

It is often convenient to use the word *term* in a broader sense than that in which it has been used in the work thus far.

For example, in the expression $(a + b) - 3(a + 2b) + 4(a + 3b)$ the binomials $(a + b)$, $-3(a + 2b)$, and $4(a + 3b)$ are often spoken of as terms, and the entire expression is then called a trinomial.

If in $(a + b)(a + b) = a^2 + 2ab + b^2$ we substitute $x + y$ for a, we get $[(x + y) + b][(x + y) + b] = (x + y)^2 + 2(x + y)b + b^2$. Expanding $(x + y)^2$ by the formula, and expanding $2b(x + y)$ also, we obtain $x^2 + 2xy + y^2 + 2bx + 2by + b^2$.

This means that if we regard $(x + y)$ as a term and apply the formula for squaring the sum of *two* terms, we can square the trinomial $x + y + b$ mentally.

Similarly
$$\begin{aligned}(x + y - b)^2 &= [(x + y) - b][(x + y) - b]\\ &= (x + y)^2 - 2b(x + y) + b^2\\ &= x^2 + 2xy + y^2 - 2bx - 2by + b^2.\end{aligned}$$

EXERCISES

Using one of the formulæ on page 93, square the following trinomials:

1. $a + b + c$.
2. $a + b - c$.
3. $a + b - 1$.
4. $a + b + 1$.
5. $a - b + c$.
6. $a - b - c$.
7. $a + 2b - 3$.
8. $a - 2b + c$.
9. $2a + b - 4c$.
10. $3a - 2b + 5c$.
11. $2a - 3b - 4c$.
12. $4c - 1 + 2ab$.

43. The product of the sum and the difference of two terms. The multiplication

$$\begin{array}{lll} a & +b & \\ a & -b & \\ \hline a^2 & +ab & \\ & -ab & -b^2 \\ \hline a^2 & & -b^2 \end{array}$$

gives the formula

$$\boldsymbol{(a + b)(a - b) = a^2 - b^2.}$$

This may be expressed in words as follows:

III. *The product of the sum and the difference of two terms equals the difference of their squares taken in the same order as the difference of the terms.*

EXERCISES

Expand by the preceding formula:

1. $(6+3)(6-3)$.
2. $(8+4)(8-4)$.
3. $(9-2)(9+2)$.
4. $(10-4)(10+4)$.
5. $(7+2)(7-2)$.
6. $(12-3)(12+3)$.
7. $22 \cdot 18$.

Solution: $22 \cdot 18 = (20+2)(20-2) = 400 - 4 = 396$.

8. $35 \cdot 25$.
9. $33 \cdot 27$.
10. $36 \cdot 44$.
11. $35 \cdot 45$.
12. $52 \cdot 48$.
13. $65 \cdot 75$.
14. $72 \cdot 68$.
15. $75 \cdot 85$.
16. $97 \cdot 103$.
17. $(x+3)(x-3)$.
18. $(a+5)(a-5)$.
19. $(2x+4)(2x-4)$.
20. $(3n+5)(3n-5)$.
21. $(x+y)(x-y)$.
22. $(x-a)(x+a)$.
23. $(x-1)(x+1)$.
24. $(a+2)(a-2)$.
25. $(a^2+3)(a^2-3)$.
26. $(4+y^2)(4-y^2)$.
27. $(4-x)(x+4)$.
28. $(2c+a)(2c-a)$.
29. $(3a+b)(b-3a)$.
30. $(4b+2c)(4b-2c)$.
31. $(3xy-2)(3xy+2)$.
32. $(4ab-3)(4ab+3)$.
33. $(a^4-b^3)(a^4+b^3)$.
34. $(a^6+a^3)(-a^3+a^6)$.
35. $(x^3-2y)(x^3+2y)$.
36. $(4ab-a^2)(4ab+a^2)$.
37. $(3cd^2+2d)(-2d+3cd^2)$.
38. $(6cd+3)(-3+6cd)$.
39. $(4xy+2y)(2y-4xy)$.
40. $(3abc-2bc)(2bc+3abc)$.

Perform the indicated division:

41. $(a^2-b^2) \div (a+b)$.
42. $(c^2-d^2) \div (c-d)$.
43. $(9-b^2) \div (3+b)$.
44. $(36-a^2) \div (6-a)$.
45. $(9x^2-16) \div (4+3x)$.
46. $(x^4-1) \div (x^2-1)$.

Find a binomial divisor for each of the following binomials:

47. $x^2 - y^2$.	**49.** $4x^2 - 9$.	**51.** $16 - x^8$.
48. $x^2 - 1$.	**50.** $25 - 16x^2$.	**52.** $y^6 - 4$.

State the two binomials whose product is:

53. $c^2 - d^2$.	**55.** $n^2 - 16$.	**57.** $36b^2 - 1$.
54. $n^2 - 4$.	**56.** $9 - 4a^2$	**58.** $a^2 - 9$.

59. $25 - 4n^2$.
60. $100 - 9x^2$.

If in $(a + b)(a - b) = a^2 - b^2$ we replace a by $x + y$, we get $[(x + y) + b][(x + y) - b] = (x + y)^2 - b^2$; which, when $(x + y)^2$ is expanded, becomes $x^2 + 2xy + y^2 - b^2$.

Similarly, replacing b by $(x + y)$, we get

$$[a + (x + y)][a - (x + y)] = a^2 - (x + y)^2.$$

Expanding, $= a^2 - (x^2 + 2xy + y^2)$.
Removing the parenthesis, $= a^2 - x^2 - 2xy - y^2$.

Perform the indicated multiplication:

61. $[(x + y) + 1][(x + y) - 1]$.
62. $(x + a + 3)(x + a - 3)$.
63. $[(x - a) + 3][(x - a) - 3]$.
64. $(x + 4 + c)(x + 4 - c)$.
65. $(2a - b + c)(2a - b - c)$.
66. $[x + (b + c)][x - (b + c)]$.
67. $[x + (b - c)][x - (b - c)]$.
68. $[3 + (x - y)][3 - (x - y)]$.
69. $[4x + (2y - x)][4x - (2y - x)]$.
70. $[10 - (a - 5)][10 + (a - 5)]$.

State the two binomials whose product is:

71. $49x^2 - 1$.	**76.** $(3y - z)^2 - b^2$.
72. $64x^2 - 25$.	**77.** $b^2 - (x + y)^2$.
73. $(a + b)^2 - 1$.	**78.** $b^2 - (x - y)^2$.
74. $(x - y)^2 - 4$.	**79.** $c^2 - (x + c)^2$.
75. $(2x - 1)^2 - a^2$.	**80.** $4 - (x - 2)^2$.

44. The product of two binomials having a common term. The multiplication

$$\begin{array}{lllll} x & +a & & & \\ x & +b & & & \\ \hline x^2 & +ax & & & \\ & +bx & +ab & & \\ \hline x^2 & +(a+b)x & +ab & & \end{array}$$

gives the formula

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

This may be expressed in words as follows:

IV. *The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.*

EXERCISES

Expand by the preceding formula:

1. $(x+1)(x+2)$.
2. $(x+2)(x+3)$.
3. $(x+3)(x+4)$.
4. $(x+4)(x+5)$.
5. $(x+5)(x+6)$.
6. $(n+1)(n-2)$.
7. $(n-2)(n+3)$.
8. $(n-3)(n+4)$.
9. $(n-4)(n+5)$.
10. $(n-5)(n+6)$.
11. $(a-1)(a-2)$.
12. $(a-2)(a-3)$.
13. $(a-3)(a-4)$.
14. $(a-4)(a-5)$.
15. $(a-4)(a-6)$.
16. $(2y+3)(2y+4)$.
17. $(2y+2)(2y+3)$.
18. $(3a+1)(3a+4)$.
19. $(2n+3)(2n+5)$.
20. $(2n+3)(2n-5)$.
21. $(2n+3)(2n-4)$.
22. $(2a+1)(2a-5)$.
23. $(3a-5)(3a+1)$.
24. $(4a+3)(4a-5)$.
25. $(4a-5)(4a-6)$.
26. $(4ab+1)(4ab-6)$.
27. $(3x-2)(3x+5)$.
28. $(4a-3b)(4a+5b)$.
29. $(5a-6b)(5a-7b)$.

Perform mentally the indicated division:

30. $\dfrac{x^2+3x+2}{x+1}$. **31.** $\dfrac{x^2+5x+6}{x+2}$. **32.** $\dfrac{x^2+4x+3}{x+3}$.

33. $\dfrac{x^2+6x+5}{x+1}$.

34. $\dfrac{x^2+7x+12}{x+3}$.

35. $\dfrac{x^2-7x+12}{x-3}$.

36. $\dfrac{x^2-5x+6}{x-3}$.

37. $\dfrac{x^2-5x+4}{x-1}$.

38. $\dfrac{x^2-6x+5}{x-5}$.

39. $\dfrac{x^2-7x+6}{x-6}$.

40. $\dfrac{x^2-7x+12}{x-4}$.

Find an exact binomial divisor for each of the following trinomials:

41. x^2+3x+2.
42. x^2+5x+6.
43. $x^2+7x+12$.
44. x^2+6x+8.
45. $x^2+7x+10$.
46. $x^2+8x+15$.
47. x^2-3x+2.
48. x^2-5x+6.
49. x^2-6x+8.
50. $x^2-8x+15$.
51. $x^2-8x+12$.
52. $x^2-9x+14$.

State the two binomials whose product is:

53. x^2+8x+7.
54. x^2+9x+8.
55. $x^2-10x+9$.
56. $x^2-11x+10$.
57. $x^2-10x+16$.
58. $x^2+10x+21$.
59. $x^2+9x+20$.
60. $x^2-9x+18$.
61. $x^2+9x+14$.
62. $x^2-12x+32$.
63. $x^2+11x+10$.

45. The square of any polynomial. The multiplication

$$\begin{array}{llllll}
a & +b & -c & & & \\
a & +b & -c & & & \\ \hline
a^2 & +ab & -ac & & & \\
 & +ab & & +b^2 & -bc & \\
 & & -ac & & -bc & +c^2 \\ \hline
a^2 & +2ab & -2ac & +b^2 & -2bc & +c^2
\end{array}$$

gives the formula

$$(a+b-c)^2 = a^2+b^2+c^2+2ab-2ac-2bc.$$

This may be expressed in words as follows:

V. *The square of any polynomial is equal to the sum of the squares of the terms, plus twice the algebraic product of each term by each term that follows it in the polynomial.*

EXERCISES

Expand by the preceding formula:

1. $(a + b + c)^2$.
2. $(a + b + 1)^2$.
3. $(a + b + 2)^2$.
4. $(2a + b - c)^2$.
5. $(a - b - c)^2$.
6. $(2x + y - 1)^2$.
7. $(4x + y - 2)^2$.
8. $(a - bc + d)^2$.
9. $(3a - 3b + 1)^2$.
10. $(a - b + c - d)^2$.
11. $(x + y - a + 1)^2$.
12. $(a - y + b - 3)^2$.

46. The cube of a binomial. The multiplication

$$\begin{array}{llll}
a + b & & & \\
\underline{a + b} & & & \\
a^2 + ab & & & \\
\underline{\quad\;\; ab + b^2} & & & \\
a^2 + 2ab + b^2 & & & \\
\underline{a + b \qquad\qquad\quad} & & & \\
a^3 + 2a^2b + ab^2 & & & \\
\underline{\qquad\quad a^2b + 2ab^2 + b^3} & & & \\
a^3 + 3a^2b + 3ab^2 + b^3 & & &
\end{array}$$

gives the formula $\boldsymbol{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$.

Similarly $\boldsymbol{(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$.

EXERCISES

Expand by the preceding formula:

1. $(x + y)^3$.
2. $(x - y)^3$.
3. $(x + 1)^3$.
4. $(x - 1)^3$.
5. $(x + 2)^3$.
6. $(x - 2)^3$.
7. $(x - 3)^3$.
8. $(x + 3)^3$.
9. $(2x + y)^3$.
10. $(x - 2y)^3$.
11. $(2x - 1)^3$.
12. $(3x + 2)^3$.
13. $(3x - 2y)^3$.
14. $(4x + 3y)^3$.
15. $(5x - 2y)^3$.

16. Express in words the formula for the cube of the binomial $a + b$.

17. Express in words the formula for the cube of the binomial $a - b$.

CHAPTER XIII

FACTORING

47. Definition. Factoring is the process of finding the two or more expressions whose product is equal to a given expression.

The subject of factoring is very extensive. In this chapter we shall consider only the more common forms of factorable expressions and only such factors as do not contain irrational numbers and fractional terms (see § 91).

Thus fractional expressions like $\frac{1}{9} - a^2$, $\frac{1}{a^2} - 4$, etc., will be regarded as prime,* though the student can readily prove that $\left(\frac{1}{3} + a\right)\left(\frac{1}{3} - a\right) = \frac{1}{9} - a^2$, and that $\left(\frac{1}{a} + 2\right)\left(\frac{1}{a} - 2\right) = \frac{1}{a^2} - 4$. Possibly he can see that $(\sqrt{3} + a)(\sqrt{3} - a) = 3 - a^2$, and perhaps that $(x + \sqrt{2}\,y)(x - \sqrt{2}\,y) = x^2 - 2\,y^2$. But here also $3 - a^2$ and $x^2 - 2\,y^2$ are considered prime because their factors contain the irrational numbers $\sqrt{3}$ and $\sqrt{2}$.

48. Roots of monomials. In factoring it is often necessary to find the square root, the cube root, and other roots of various monomials.

The **square root** of a monomial is one of the two equal factors whose product is the monomial.

Since $+2 \cdot +2 = 4$ and $-2 \cdot -2 = 4$, the square root of 4 is ± 2, which means plus 2 or minus 2.

Similarly the square root of 9 is ± 3 and the square root of a^2 is $\pm a$.

That is, *Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.*

It is customary to speak of the positive square root of a number as the **principal square root,** and if no sign precedes the radical the principal root is understood. When both the positive and the negative square roots are considered, both signs must precede the radical.

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* An integral expression will be considered **prime** when no two rational integral expressions (see § 57) can be found (except the expression itself and 1) whose product is the given expression.

Thus $\sqrt{4}=2$, not -2; $-\sqrt{4}=-2$, not $+2$.
Since $a^5\cdot a^5=(-a^5)(-a^5)=a^{10}$, $\pm\sqrt{a^{10}}=\pm a^5$.
Similarly $a^6b^3\cdot a^6b^3=a^{12}b^6$, and $\pm\sqrt{a^{12}b^6}=\pm a^6b^3$.

That is, *The exponent of any letter in the square root of a monomial is one half the exponent of that letter in the monomial.*

Hence for extracting the square root of a monomial we have the

RULE. *Write the square root of the numerical coefficient preceded by the double sign ± and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule much like the preceding one holds for fourth root, sixth root, and other even roots.

Thus $\pm\sqrt[4]{81\,a^4}=\pm 3\,a$, and $\pm\sqrt[6]{x^{12}}=\pm x^2$.

In the chapters on Factoring and Fractions where square roots arise only the *positive* square root will be considered.

According to the definition of square root the two factors of a term, either of which is its square root, *must be equal.* Consequently they must have the same sign. Since the product of two terms having like signs cannot be negative, we cannot extract the square root of a negative term. Hence we do not consider the square root of such terms as -4, $-9\,a^2$, and $-16\,x^2y^4$ in this chapter.

The **cube root** of a monomial is one of the three equal factors whose product is the monomial.

In this chapter only a single cube root of a number is considered; that is, the **principal cube root.**

Since $3\cdot 3\cdot 3=27$, $\sqrt[3]{27}=3$.
And as $-3\cdot -3\cdot -3=-27$, $\sqrt[3]{-27}=-3$.

That is, *The cube root of a monomial has the same sign as the monomial.*

Since $a^4 \cdot a^4 \cdot a^4 = a^{12}$, $\sqrt[3]{a^{12}} = a^4$.

Similarly $a^2b^3 \cdot a^2b^3 \cdot a^2b^3 = a^6b^9$, and $\sqrt[3]{a^6b^9} = a^2b^3$.

That is, *The exponent of any letter in the cube root of a term is one third of the exponent of that letter in the term.*

Hence for extracting the cube root of a monomial we have the

Rule. *Write the cube root of the numerical coefficient preceded by the sign of the monomial and followed by all the letters of the monomial, giving to each letter an exponent equal to one third of its exponent in the monomial.*

A rule much like the preceding one holds for fifth root, seventh root, and other odd roots.

Thus $\sqrt[3]{-8} = -2$, $\sqrt[5]{x^{10}} = x^2$, and $\sqrt[7]{128\,x^{21}} = 2\,x^3$.

EXERCISES

Find the value of the following:

1. $\sqrt{4\,a^2}$.
2. $\sqrt{9\,a^4}$.
3. $\sqrt{25\,a^6x^4}$.
4. $\sqrt{4\,x^4}$.
5. $\sqrt{9\,x^6}$.
6. $\sqrt{16\,x^{16}}$.
7. $\sqrt{49\,x^{14}y^{10}}$.
8. $\sqrt{81\,x^{12}d^{20}}$.
9. $\sqrt{121\,c^{22}d^6}$.
10. $\sqrt{169\,x^6}$.
11. $\sqrt{196\,y^{10}}$.
12. $\sqrt{225\,y^4z^6}$.
13. $\sqrt{256\,c^{12}}$.
14. $\sqrt{400\,a^2b^2c^4}$.
15. $\sqrt{324\,b^8}$.
16. $\sqrt{625\,b^2y^{18}z^6}$.
17. $\sqrt{x^{2n}}$.
18. $\sqrt{x^{4n}}$.
19. $\sqrt{x^{6n}}$.
20. $\sqrt{a^{4n}b^{8n}}$.
21. $\sqrt[3]{8}$.
22. $\sqrt[3]{-8\,a^3}$.
23. $\sqrt[3]{27}$.
24. $\sqrt[3]{64}$.
25. $\sqrt[3]{-125}$.
26. $\sqrt[3]{216}$.
27. $\sqrt[3]{-a^3}$.
28. $\sqrt[3]{-a^6}$.
29. $\sqrt[3]{a^9b^3}$.
30. $\sqrt[3]{343\,y^3}$.
31. $\sqrt[3]{512\,x^{15}}$.
32. $\sqrt[3]{-729\,a^3b^9}$.
33. $\sqrt[3]{1000\,a^6b^{12}}$.
34. $\sqrt[3]{-27\,a^3b^6c^{12}}$.
35. $\sqrt[3]{8\,a^{12}b^3c^9}$.
36. $\sqrt[3]{-125\,a^6b^6z^6}$.
37. $\sqrt[3]{x^{3n}}$.
38. $\sqrt[3]{x^{6n}}$.
39. $\sqrt[3]{-x^{9n}}$.
40. $\sqrt[3]{x^{12n}y^{6n}}$.
41. $\sqrt[5]{x^{15}}$.
42. $\sqrt[5]{32\,x^{10}}$.

49. Polynomials with a common monomial factor. The type form is

$$ab + ac - ad.$$

Plainly $ab + ac - ad = a(b + c - d)$.

This gives, for factoring expressions having a common monomial factor, the

RULE. *Determine by inspection the greatest monomial factor which occurs in each term of the polynomial.*

Divide the polynomial by this monomial factor.

Write the quotient in a parenthesis preceded by the monomial factor.

Example. Factor $9x^2y - 36y^2$.

Solution: By inspection the greatest monomial factor of each term is $9y$. Dividing the binomial by $9y$, the quotient is $x^2 - 4y$.

Therefore $9x^2y - 36y^2 = 9y(x^2 - 4y)$.

EXERCISES

Factor the following:

1. $3x + 6$.
2. $x^3 - x^2$.
3. $8x - 2x^4$.
4. $xy + y^2$.
5. $5b^3 - 15b^6$.
6. $c^3 - c^2y$.
7. $x^3 - x + x^2$.
8. $c^5 - c^3 + c^2$.
9. $3y - 9y^4 + 12$.
10. $c^2 + 2bc - c$.
11. $3y^2 - 15y + 6y^3$.
12. $10ab - 14bc - 8b^2$.
13. $y^5 + 3y^4z + 6y^3z^2$.
14. $4a^2x^2 - 6a^3y^3 + 12a^4x^2y^2$.
15. $12z^3 + 30a^4z^7 - 18c^3z^5$.
16. $6c^6 + 10c^{10} - 20c^2$.
17. $8y^8 - 4y^4 - 12y^{12}$.
18. $x^4 + x^3 - x^2 + x$.
19. $-8y^6 - 4y^6 + 12y^3 + 6y^5$.
20. $14b^5 - 49b^7 + 21b^2 - 7b^3$.
21. $-a^2x^2y^3 - 3a^4x^3y^2 - 2a^3x^5y^6z^3 + 5a^4x^4y^4z$.
22. $-18a^3b^3c^4 - 45a^4cy^2 - 36a^3c^2xz - 63a^4cxy$.
23. $32c^7x^2 + 80c^{10}bx - 112c^6b + 48c^{12}x$.
24. $56a^4b^3x^2 - 28a^3x + 112a^5x^2y^2 - 196a^6x^3$.

50. Polynomials which may be factored by grouping terms. The type form is

$$ax + ay + bx + by.$$

Plainly $ax + ay + bx + by = a(x + y) + b(x + y)$.

Dividing both terms of $a(x + y) + b(x + y)$ by $(x + y)$, the quotient is $a + b$.

Therefore $ax + ay + bx + by = (x + y)(a + b)$.

The preceding example illustrates the

Rule. *Arrange the terms of the polynomial to be factored, in groups of two or more terms each, such that in each group a monomial factor may be written outside a parenthesis, which in each case contains the same expression.*

Then divide by the expression in parenthesis and write the divisor as one factor and the quotient as the other.

Polynomials which may be factored by grouping terms, according to the foregoing rule, usually contain either four, six, or eight terms.

It is important to note that one can obtain two apparently different sets of factors for a given expression. Thus

$$(a - 3b)(c - d) = (3b - a)(d - c) = ac - 3bc - ad + 3bd.$$

But the difference between the first pair of binomials and the second pair is only one of sign, and it is customary in this and in similar cases to regard either pair of binomials as a different form of the other pair.

The relation that the process of factoring bears to the processes of multiplication and division of polynomials should be constantly kept in mind. In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience is supposed to enable us to determine the factors. For this reason a very careful study of several forms of products is necessary.

There is no simple operation the performance of which makes us sure that we have found the *prime* factors of a given expression. Only insight and experience enable us to find prime factors with certainty.

A partial check, however, that may be applied to all the exercises in factoring consists in actually multiplying together the factors that have been found. The result should be the original expression.

EXERCISES

Factor the following (see Exercises 23–31, page 32):

1. $2(x+y)+a(x+y)$.

2. $b(x+2y)+(x+2y)$.

3. $c(c-a)-a(c-a)$.

4. $2c(3b-d)+3a(3b-d)$.

5. $6d(c+x)-(c+x)+4k(c+x)$.

6. $-5(2a-3b)+7a(2a-3b)-3b(2a-3b)$.

7. $k(a-b)+3(b-a)$.

This can be written $k(a-b)-3(a-b)$.

8. $h(x-y)+k(y-x)$.

9. $5x(c-2d)-6(2d-c)$.

10. $3y(k-4h)+(4h-k)$.

11. $2c(4k-3c)-5h(3c-4k)$.

12. $3y(5x-b)-7(b-5x)+6z(5x-b)$.

13. $ab+bx+ac+cx$.

Solution:
$$\begin{aligned} ab+bx+ac+cx &= (ab+bx)+(ac+cx) \\ &= b(a+x)+c(a+x) \\ &= (a+x)(b+c). \end{aligned}$$

14. $ax-bx+a-b$.

Solution:
$$\begin{aligned} ax-bx+a-b &= (ax-bx)+(a-b) \\ &= x(a-b)+1(a-b) \\ &= (a-b)(x+1). \end{aligned}$$

15. $4x^3 - 3xy - 8x^2y + 6y^2$.

Solution: $4x^3 - 3xy - 8x^2y + 6y^2$

$$\begin{aligned} &= (4x^3 - 3xy) + (-8x^2y + 6y^2) \\ &= x(4x^2 - 3y) + 2y(-4x^2 + 3y) \\ &= x(4x^2 - 3y) - 2y(4x^2 - 3y) \\ &= (4x^2 - 3y)(x - 2y). \end{aligned}$$

Expressions similar to 13, 14, and 15 may be grouped in more than one way. For example, the terms of 15 can be grouped thus:

$$\begin{aligned} &(4x^3 - 8x^2y) + (-3xy + 6y^2) \\ &\qquad = 4x^2(x - 2y) + 3y(-x + 2y) \\ &\qquad = 4x^2(x - 2y) - 3y(x - 2y) \\ &\qquad = (x - 2y)(4x^2 - 3y). \end{aligned}$$

16. $ax + bx + ay + by$.

17. $ch + h^2 + cx + hx$.

18. $bh - ch + bk - ck$.

19. $bx + xz - ab - az$.

20. $ax + 3a - bx - 3b$.

21. $2x^2 + 10xy - 4x - 20y$.

22. $6ab - 2ac + 3b - c$.

23. $36ax + 45ac - 4x - 5c$.

24. $6a^3 + 14ab - 15a^2y - 35by$.

25. $3ax - ay - 6bx + 2by$.

26. $12ax - 6ay - 50cx + 25cy$.

27. $2h^3 - 3h^2k - 10h + 15k$.

28. $10dh - 45hk - 22cd + 99ck$.

29. $28hx + 9ky - 21hy - 12kx$.

30. $-2ax + 7a^2 + 16bx - 56ab$.

31. $6hk + 15xy - 10ky - 9hx$.

32. $-21ax + 12cx - 4cd + 7ad$.

33. $7gy - 77hy - cg + 11ch$.

34. $5a^3 + 10a - 5a^2 - 10$.

35. $3a - 5ax^3 - 6ax + 10ax^4$.

36. $4abxy - 24dxy - 3abgh + 18dgh$.

37. $8acxy - 20bx^2y - 6abc^2 + 15b^2cx$.

38. $ax + bx + cx + ay + by + cy$.

39. $cg + 2ch + ck - gx - 2hx - kx$.

40. $3\,ab - bx + 2\,by - 3\,ac + cx - 2\,cy$.

41. $4\,ax - ac + 2\,ay + 12\,bx - 3\,bc + 6\,by$.

42. $ax - 6\,cy - 2\,bx + 4\,by + 3\,cx - 2\,ay$.

43. $-2\,xz + 6\,ay + cz + 4\,xy - 3\,az - 2\,cy$.

51. Trinomials which are perfect squares. The type form is

$$a^2 \pm 2\,ab + b^2.$$

This, by § 42, gives us the two expressions:

$$a^2 + 2\,ab + b^2 = (a + b)^2,$$
$$a^2 - 2\,ab + b^2 = (a - b)^2.$$

If an algebraic expression is the product of two equal factors, it is said to be a **perfect square.**

A trinomial, arranged according to the descending powers of one letter, *is a perfect square if the absolute value of the middle term is twice the product of the absolute values of the square roots of the other two terms.*

Thus in the type form above, $2\,ab = 2 \cdot \sqrt{a^2} \cdot \sqrt{b^2}$.

Similarly the trinomial $4\,x^2 - 20\,xy^2 + 25\,y^4$ is a perfect square, since $20\,xy^2 = 2 \cdot \sqrt{4\,x^2} \cdot \sqrt{25\,y^4} = 2 \cdot 2\,x \cdot 5\,y^2$.

EXERCISES

Form perfect trinomial squares of the following by supplying the missing term:

1. $c^2 + (?) + d^2$.

2. $a^2 + (?) + 4$.

3. $4\,a^2 + (?) + 1$.

4. $x^6 + (?) + 9$.

5. $4\,x^2 + (?) + 25$.

6. $9\,x^6 + (?) + 4\,y^2$.

7. $100\,x^2 + (?) + 36\,y^2$.

8. $49 + (?) + 64\,y^8$.

9. $x^2 + 2\,x + (?)$.

10. $y^2 - 6\,y + (?)$.

11. $4\,y^2 - 40\,y + (?)$.

12. $9\,y^2 + 36\,y + (?)$.

13. $16\,z^4 - 16\,xz^2 + (?)$.

14. $121\,y^6 - 88\,y^3z + (?)$.

15. $100\,a^8 + 240\,a^4x^5 + (?)$.

16. $(?) - 6\,y^2 + 9$.

17. $(?) - 10\,y^4 + 25$.

18. $(?) + 16\,k^2 + 4$.

19. $(?) - 80\,k^2h^2 + 64\,h^4$.

20. $(?) + 104\,a^8k + 169\,k^2$.

For obtaining *one* of the two equal factors of a perfect trinomial square, we have the

Rule. *Arrange the terms of the trinomial according to the descending powers of some letter in it.*

Extract the square root of the first and third terms and connect the results by the sign of the middle term.

EXERCISES

Factor the following:

1. $c^2 + 2cd + d^2$.
2. $x^2 + 2xy + y^2$.
3. $x^2 + 2x + 1$.
4. $a^2 - 10a + 25$.
5. $a^2 + 8a + 16$.
6. $4h^2 + 4h + 1$.
7. $9 - 6x + x^2$.
8. $16k^2 + 1 - 8k$.
9. $4h^2 - 12h + 9$.
10. $x^4 + 1 + 2x^2$.
11. $x^6 - 6x^3 + 9$.
12. $25h^2 + 4k^2 - 20hk$.
13. $9h^2 - 24hk + 16k^2$.
14. $x^2 - 24xz^2 + 144z^4$.
15. $12x^4z^3 + 9z^6 + 4x^8$.
16. $c^6d^{10} + 2c^3d^5 + 1$.
17. $24x^2y^2 + 16y^4 + 9x^4$.
18. $x^4 + 4y^4 + 4x^2y^2$.
19. $4x^2y^8 - 4xy^4 + 1$.
20. $25h^6k^2 - 30h^3kz + 9z^2$.
21. $121c^6d^4 - 220c^3d^2g^2 + 100g^4$.
22. $169a^4 - 156a^2x^4y^3 + 36x^8y^6$.

It is only in the beginning of factoring that polynomials are classified for the student. In the practical work of handling fractions and solving equations he must determine for himself the type of the polynomial to be factored. It is therefore very important that he fix in mind the various types and the manner of factoring each. Moreover, he should remember that the polynomials which arise in practice often have three or more factors. Miscellaneous review exercises afford excellent practice in recognizing types and in determining all the prime factors.

The suggestions given on page 120 will prove helpful, though only the first three of the types there given have as yet been considered.

REVIEW EXERCISES

Separate into prime factors:

1. $x^3 + 2x^2 + x.$

2. $x^5 - 4x^4 + 4x^3.$

3. $2a^4 + 12a^2 + 18.$

4. $3a^2 + 18a + 27.$

5. $x^6 - 20x^4 + 100x^2.$

6. $5cx^6 - 70cx^3 + 245c.$

7. $16x^8 - 24x^6 + 9x^4.$

8. $28x^2 - 28x^3 + 7x^4.$

9. $3ax + 3ay + 3bx + 3by.$

10. $4cx - 4cy + 4dx - 4dy.$

11. $ax - a + x - 1.$

12. $63xy - 84y^2 + 98yz.$

13. $30ax - 34bx - 15a + 17b.$

14. $ax - bx + cx + ay - by + cy.$

15. $14anx - 21bny - 7n.$

16. $3ax - 5by - 5ay + 3bx.$

17. $56a^2 - 40ab + 63ac - 45bc.$

52. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

Suppose $\quad (x + h)(x + k) \equiv x^2 + bx + c.$

Then $\quad x^2 + (h + k)x + hk \equiv x^2 + bx + c.$

Since this is an *identity* we may assume that the corresponding terms in each member are equal.

That is, $\quad (h + k)x = bx,$

or $\quad h + k = b;$

and $\quad hk = c.$

Therefore, if two numbers exist whose sum is b and whose product is c, the trinomial $x^2 + bx + c$ can be factored; if such numbers do not exist, the trinomial cannot be factored. Many trinomials of the form $x^2 + bx + c$ are prime; for factoring such as are not we have the

RULE. *Find two numbers whose algebraic product is* **c** *and whose algebraic sum is* **b**.

Write for the factors the two binomials which have **x** *for the common term and the numbers just obtained for the unlike terms.*

EXAMPLES

1. Factor $x^2 + 7x + 12$.

Solution: Here it is necessary to find two numbers whose product is $+12$ and whose sum is $+7$. Now $12 = 1 \cdot 12$, or $2 \cdot 6$, or $3 \cdot 4$. Obviously only the last pair gives 7 for a sum.

Therefore $$x^2 + 7x + 12 = (x+3)(x+4).$$

2. Factor $x^2 - 12x + 32$.

Solution: Since 32 is positive its two factors must have the same sign; since -12 is negative both factors must be negative. Now $32 = 1 \cdot 32$, or $2 \cdot 16$, or $4 \cdot 8$. By inspection of these products, -4 and -8 are found to be the required numbers.

Therefore $$x^2 - 12x + 32 = (x-4)(x-8).$$

3. Factor $x^2 - 6x - 40$.

Solution: The product, -40, is negative; hence the required factors have *unlike* signs. The sum, -6, being negative, the negative factor of -40 must have the greater absolute value. Now $40 = 1 \cdot 40$, or $2 \cdot 20$, or $4 \cdot 10$, or $5 \cdot 8$. We see that $4 + (-10) = -6$.

Therefore $$x^2 - 6x - 40 = (x+4)(x-10).$$

4. Factor $x^2 + 13x - 48$.

Solution: The product is negative; hence the required factors have unlike signs. The sum being positive, the positive factor of 48 must have the greater absolute value. Now $48 = 1 \cdot 48 = 2 \cdot 24 = 3 \cdot 16 = 4 \cdot 12 = 6 \cdot 8$. We observe that $16 + (-3) = 13$.

Therefore $$x^2 + 13x - 48 = (x-3)(x+16).$$

EXERCISES

Factor:

1. $x^2 + 5x + 6$.

2. $x^2 + 9x + 20$.

3. $x^2 - 7x + 12$.

4. $x^2 - 9x + 14$.

5. $x^2 + 19x + 18$.

6. $x^2 - 18x + 17$.

7. $x^2 - x - 6$.

8. $x^2 - 3x - 40$.

9. $x^2 + x - 30$.

10. $x^2 - 2x - 35$.

11. $x^2 + 11x - 42$.

12. $6 - 5x - x^2$.

Solution: Arranging the terms, this becomes $-x^2 - 5x + 6$. The minus sign in the term $-x^2$ prevents our obtaining its square root. Taking out the common factor -1, and factoring as usual,

$$-x^2 - 5x + 6 = (-1)(x^2 + 5x - 6) = (-1)(x-1)(x+6)$$
$$= (-x+1)(x+6) = (1-x)(x+6).$$

13. $14 - 5x - x^2$.

14. $21 + 4x - x^2$.

15. $24 + 10a - a^2$.

16. $10 + 3x - x^2$.

17. $72 + c - c^2$.

18. $x^2 + x - 90$.

19. $28 + x^2 - 11x$.

20. $27 - 12x + x^2$.

21. $8x + x^2 - 48$.

22. $x^2 - x - 90$.

23. $x^2 - 5ax + 6a^2$.

24. $x^2 + 7ax + 12a^2$.

25. $a^2x^2 - 2ax - 15$.

26. $h^2k^2 - 9hk + 20$.

27. $x^4 - 5x^2 - 14$.

28. $x^2 + x^4 - 110$.

29. $x^6 - 11x^3 + 18$.

30. $b^2x^2 - 3bxy - 28y^2$.

31. $h^2k^2 - 5hkx - 36x^2$.

32. $a^6b^2 - 5a^3bx - 24x^2$.

33. $h^4d^2 + 10h^2dg^3 - 24g^6$.

34. $9k^2x^4y^6 + k^4x^8 - 22y^{12}$.

REVIEW EXERCISES

Factor:

1. $x^4 + 6x^3 + 9x^2$.
2. $x^3 - 5x^2 + 6x$.
3. $3b^2 - 9b - 30$.
4. $3ax^3 - 3ax^2 - 6ax$.
5. $4x^5 - 4x^3 - 360x$.
6. $3x^4 - 21x^3 - 54x^2$.
7. $4a^6y^4 - 64a^4y^2 + 256a^2$.
8. $3x - 3ax - 18a^2x$.
9. $56x + x^2 - x^3$.
10. $140h^2 + 23h^3 - h^4$.
11. $3x^3 - 3x^4 + 60x^2$.
12. $32a^2x^2 + 2a^2 - 16a^2x$.
13. $4ay^2x^2 - 40ay^2x + 100ay^2$.
14. $3x^4 - 3x^3 + 3x^2 - 3x$.
15. $2ax^3 - 4ax^2 - 12ax + 24a$.
16. $5xyz - 15yz + 10xz - 30z$.

53. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

A trinomial of this type can always be factored by grouping, if two numbers can be found whose product is ac and whose algebraic sum is b. If such numbers do not exist, the trinomial is prime.

EXAMPLES

1. Factor $6x^2 + 17x + 7$.

Solution: Here $b = 17$ and $ac = 6 \cdot 7 = 42$. Now $42 = 1 \cdot 42 = 2 \cdot 21 = 3 \cdot 14 = 6 \cdot 7$. Obviously the only pair of factors whose sum is 17 is 3 and 14.

Therefore we write
$$\begin{aligned} 6x^2 + 17x + 7 &= 6x^2 + 3x + 14x + 7 \\ &= 3x(2x+1) + 7(2x+1) \\ &= (2x+1)(3x+7). \end{aligned}$$

2. Factor $3x^2 + 10x - 8$.

Solution: Here $b = 10$ and $ac = 3(-8) = -24$. Now $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$. Since the product, -24, is negative the parts into which $+10$ is to be separated must have opposite signs. Obviously $+12$ and -2 are the required numbers.

Therefore we write
$$\begin{aligned} 3x^2 + 10x - 8 &= 3x^2 + 12x - 2x - 8 \\ &= 3x(x+4) - 2(x+4) \\ &= (x+4)(3x-2). \end{aligned}$$

For factoring a trinomial of the form $ax^2 + bx + c$ the preceding examples illustrate the

RULE. *Find two numbers whose product is* **ac** *and whose algebraic sum is* **b**.

Replace **bx** *by two terms in* **x** *whose respective coefficients are the numbers just found, and factor by grouping terms.*

A perfect trinomial square and a trinomial of the form $x^2 + bx + c$ are special forms of $ax^2 + bx + c$. Therefore the exercises on page 109 and those on pages 111–112 could be factored by the rule just given.

EXERCISES

Factor:

1. $2x^2 + 5x + 3$.

2. $3x^2 + 5x + 2$.

3. $2x^2 - 7x + 3$.

4. $x^2 + 6x + 9$.

5. $3x^2 + 8x + 4$.

6. $5x^2 - 8x + 3$.

7. $9x^2 + 18x + 8$.

8. $7x^2 + 9ax + 2a^2$.

9. $4x^2 + 16x + 15$.

10. $2x^2 - 3x - 2$.

11. $2x^2 - x - 3$.

12. $4x^2 - 7x - 15$.

13. $3x^4 + 7x^2 - 6$.
14. $3x^2 + 5xy - 2y^2$.
15. $3x^2 - x - 2$.
16. $2x^2 - 5x - 3$.
17. $2x^2 + xz - 3z^2$.
18. $6x^2 - x - 2$.
19. $25x^2 - 20x + 3$.
20. $49x^2 - 21x + 2$.
21. $36x^2 - 36x + 5$.
22. $6x^4 + 13x^2 + 6$.
23. $9x^6 - 6x^3 - 8$.
24. $5x^2 + 2x - 16$.
25. $3x^2 - 11x - 20$.
26. $5x^4 + 18x^2y + 16y^2$.
27. $6x^2 + 23x - 55$.
28. $6x^6 + 23x^3 + 21$.
29. $10x^2 - 7x - 12$.
30. $4c^2 + 11cg - 3g^2$.
31. $8x^2 + 26xy + 15y^2$.
32. $35x^2 + 22x + 3$.
33. $14x^2 - x - 3$.
34. $21x^4 - x^2 - 2$.
35. $22x^2 - 3x - 7$.
36. $18x^2 + 65x + 7$.
37. $26x^2 + 9x - 2$.
38. $14x^2 - 39x + 10$.
39. $35x^2 - 39x - 36$.
40. $42x^2 - 9x - 6$.

REVIEW EXERCISES

Factor:

1. $4ax^2 + 8ax + 3a$.
2. $2c^2x^2 + c^2x - 3c^2$.
3. $48ax^4 - 120ax^2 + 75a$.
4. $18c^2d - 30cd - 28d$.
5. $x^4 + x^3 - 110x^2$.
6. $75x^3 + 60x^2 + 12x$.
7. $4x^2y^2 - 4xy^2 - 120y^2$.
8. $70x^3 - 85x^2y - 30xy^2$.
9. $18x^3 - 39x^2 + 18x$.
10. $36x^4 - 6x^3 - 12x^2$.
11. $4x^4 - 20x^3 - 24x^2$.
12. $5ax^2 + 5ax + 5a$.

54. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

By § 43 $\qquad a^2 - b^2 = (a + b)(a - b).$

Hence the

RULE. *Regarding each term of the binomial as positive, extract its square root.*

Add the two square roots for one factor, and subtract the second from the first for the other.

EXAMPLES

1. Factor $x^8 - y^4$.

Solution: $x^8 - y^4 = (x^4 + y^2)(x^4 - y^2)$, by application of the rule.
$= (x^4 + y^2)(x^2 + y)(x^2 - y)$, by application of the rule to $x^4 - y^2$.

2. Factor $256\,x^{16} - y^8$.

Solution:
$$\begin{aligned}256\,x^{16} - y^8 &= (16\,x^8 + y^4)(16\,x^8 - y^4)\\ &= (16\,x^8 + y^4)(4\,x^4 + y^2)(4\,x^4 - y^2)\\ &= (16\,x^8 + y^4)(4\,x^4 + y^2)(2\,x^2 + y)(2\,x^2 - y).\end{aligned}$$

EXERCISES

Factor:

1. $k^2 - h^2$.
2. $h^2 - 1$.
3. $k^2 - 4$.
4. $a^4 - x^2$.
5. $1 - a^2$.
6. $1 - 4\,a^2$.
7. $4\,a^2 - 9$.
8. $25 - x^2$.
9. $h^4 - 81$.
10. $16\,x^4 - 25$.
11. $36\,c^8 - 49\,d^8$.
12. $a^2b^4 - 121$.
13. $100 - z^8$.
14. $x^4 - y^4$.
15. $64\,h^6 - 169$.
16. $a^8 - b^8$.
17. $1 - 16\,x^4$.
18. $16\,x^4 - y^8$.
19. $x^{16} - y^{16}$.
20. $x^{16} - 4\,y^{16}$.
21. $144\,a^2b^4 - 9$.
22. $196 - x^4y^6$.
23. $225\,a^4 - 16\,b^8c^{12}$.
24. $x^{2n} - y^2$.
25. $x^2 - y^{4n}$.
26. $a^{4n} - b^2$.
27. $a^{6n} - b^{2n}$.
28. $a^{4n} - b^{6n}$.
29. $4\,x^{2n} - y^{4n}$.
30. $9\,x^{2n} - 4\,y^{6n}$.
31. $(a + b)^2 - 4$.
32. $(x - y)^2 - 4\,c^2$.
33. $(a - 2\,b)^2 - 9\,x^2$.
34. $4(x + y)^2 - 1$.
35. $25(3\,b - c)^2 - 64$.

36. $4 - (a + b)^2$.

Solution:
$$\begin{aligned}4 - (a + b)^2 &= [2 + (a + b)][2 - (a + b)]\\ &= (2 + a + b)(2 - a - b).\end{aligned}$$

37. $81 - (x - y)^2$.
38. $4\,a^2 - (2\,x + y)^2$.

Some polynomials of four or six terms may be arranged as the difference of two squares and factored as in the preceding exercises.

EXAMPLES

1. Factor: $a^2 + 2ab + b^2 - c^2$.

Solution: $a^2 + 2ab + b^2 - c^2 = (a + b)^2 - c^2$
$= (a + b + c)(a + b - c)$.

2. Factor: $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$.

Solution: $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$
$= a^2 + 2ab + b^2 - (c^2 + 2cd + d^2)$
$= (a + b)^2 - (c + d)^2$
$= (a + b + c + d)(a + b - c - d)$.

EXERCISES

Factor:

1. $(a - b)^2 - 9x^2$.
2. $16y^2 - (h + 2k)^2$.
3. $25a^4 - (2b - 3c)^2$.
4. $49x^6 - (7c - 2d)^2$.
5. $1 - (5h - 3k)^2$.
6. $(5x + 2y)^2 - (3x - 7y)^2$.

Solution: $(5x + 2y)^2 - (3x - 7y)^2$
$= [(5x + 2y) + (3x - 7y)][(5x + 2y) - (3x - 7y)]$
$= (5x + 2y + 3x - 7y)(5x + 2y - 3x + 7y)$
$= (8x - 5y)(2x + 9y)$.

7. $(a - b)^2 - (a + b)^2$.
8. $(a - b)^2 - (x + y)^2$.
9. $(a + 2b)^2 - (a - 3b)^2$.
10. $(2a - 3b)^2 - (3a - 2b)^2$.
11. $(5h + 3k)^2 - (2c - 9d)^2$.
12. $(4b - a)^2 - (7a - 6b)^2$.
13. $4(a - 2b)^2 - (2x - y)^2$.
14. $9(a + b)^2 - (2a - 5b)^2$.
15. $(a - 2b)^2 - 4(a + b)^2$.
16. $y^2 - c^2 + x^2 - 2xy$.

Solution: $y^2 - c^2 + x^2 - 2xy = x^2 - 2xy + y^2 - c^2$
$= (x - y)^2 - c^2$
$= (x - y + c)(x - y - c)$.

17. $x^2 + 2ax + a^2 - y^2$.
18. $x^2 + 2x + 1 - 4z^2$.
19. $c^2 - 2cd + d^2 - 16a^2$.
20. $4x^2 - 12cx + 9c^2 - 25y^2$.
21. $x^2 - y^2 - 2ax + a^2$.
22. $k^2 - g^4 - 4kh + 4h^2$.
23. $6xy - c^2 + 9x^2 + y^2$.
24. $12ab - 4h^2 + 4b^2 + 9a^2$.
25. $1 - 4ax + 4a^2x^2 - x^2$.
26. $4c^2 - 20cd + 25d^2 - 9d^4$.

27. $9d^2 - 25a^2 - 6cd + c^2.$

28. $12ab + x^2 - 4a^2 - 9b^2.$

Solution: $12ab + x^2 - 4a^2 - 9b^2$

$$\begin{aligned} &= x^2 - 4a^2 + 12ab - 9b^2 \\ &= x^2 - (4a^2 - 12ab + 9b^2) \\ &= x^2 - (2a - 3b)^2 \\ &= (x + 2a - 3b)(x - 2a + 3b). \end{aligned}$$

29. $a^2 - b^2 - 2bc - c^2.$

30. $x^2 - a^2 - 2ac - c^2.$

31. $y^2 - b^2 + 4bc - 4c^2.$

32. $2bc - c^2 - b^2 + a^4.$

33. $9x^2 - 4y^2 - a^2 - 4ay.$

34. $6x + 9y^2 - 9 - x^2.$

35. $4bc + 1 - 4c^2 - b^2.$

36. $4bc - 4b^2 + 4x^6 - c^2.$

37. $x^2 - a^2 + y^2 - 4 - 2xy + 4a.$

Solution: $x^2 - a^2 + y^2 - 4 - 2xy + 4a$

$$\begin{aligned} &= x^2 - 2xy + y^2 - a^2 + 4a - 4 \\ &= (x^2 - 2xy + y^2) - (a^2 - 4a + 4) \\ &= (x - y)^2 - (a - 2)^2 \\ &= (x - y + a - 2)(x - y - a + 2). \end{aligned}$$

38. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2.$

39. $9k^2 - 6hk + h^2 - 4c^2 - 4cd - d^2.$

40. $x^2 - 1 + y^2 - a^2 + 2xy + 2a.$

41. $1 + 2bc + 2a - c^2 - b^2 + a^2.$

REVIEW EXERCISES

Factor:

1. $x^3 - x.$

2. $x^4 - 2x^2 + 1.$

3. $x^8 - 2x^4 + 1.$

4. $x^5 - 8x^3 + 16x.$

5. $x^5 - x.$

6. $x^{10} - x^2.$

7. $x^4 - 10x^2 + 9.$

8. $x^4 - 13x^2 + 36.$

9. $3a^2x^4 - 12a^4x^2 + 12a^6.$

10. $18a^2x^2 - 24a^2x - 10a^2.$

11. $3x^4 - 15x^2 + 12.$

12. $12a - 39ay - 51ay^2.$

13. $4n^6 + 48n^2 - 28n^4.$

14. $16x^4 + 8x^2 - 3.$

15. $a^3 - a + a^2b - b.$

16. $x^3 - x^2 - 4x + 4.$

17. $3a^3 + 3a^2 - 27a - 27.$

18. $2a^3b + 3a^2b - 8ab - 12b.$

19. $4a^2 - a^4 + 81 + 10a^2x - 36a - 25x^2.$

20. $12cd^3 - 6a^3x - a^6 + 4c^2 + 9d^6 - 9x^2.$

55. A binomial the sum or the difference of two cubes. The type form is

$$a^3 \pm b^3.$$

$a^3 + b^3$ divided by $(a + b)$ gives the quotient $a^2 - ab + b^2$, and $a^3 - b^3$ divided by $(a - b)$ gives the quotient $a^2 + ab + b^2$.

Therefore $\quad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,

and $\quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

EXAMPLES

1. Factor $x^3 + 8$.

Solution: $x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - x \cdot 2 + 2^2)$

$= (x + 2)(x^2 - 2x + 4)$.

2. Factor $27x^3 - 1$.

Solution: $27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)[(3x)^2 + 3x \cdot 1 + 1^2]$

$= (3x - 1)(9x^2 + 3x + 1)$.

3. Factor $64x^3 + 125y^6$.

Solution: $64x^3 + 125y^6 = (4x)^3 + (5y^2)^3$

$= (4x + 5y^2)[(4x)^2 - (4x)(5y^2) + (5y^2)^2]$

$= (4x + 5y^2)(16x^2 - 20xy^2 + 25y^4)$.

EXERCISES

Factor:

1. $x^3 + 1$.
2. $x^3 - 1$.
3. $a^3 - 8$.
4. $27 + b^3$.
5. $64x^3 - 1$.
6. $x^3 - y^6$.
7. $x^3 + 8y^3$.
8. $x^3 + y^9$.
9. $8x^3 - 27y^3$.
10. $27a^3 + 64b^3$.
11. $125y^3 + 8x^3$.
12. $216 - 27x^3$.
13. $x^6 + y^9$.
14. $a^6 + b^6$.
15. $a^6 - b^6$.

Hint. This expression may be regarded as the difference of two cubes, $(a^2)^3 - (b^2)^3$, or as the difference of two squares, $(a^3)^2 - (b^3)^2$. *Whenever an expression may be regarded in both these ways the latter is always preferable.* Thus $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$, etc.

16. $x^6 - 64$.
17. $64x^6 + 1$.
18. $1 - 64x^6$.
19. $a^6b^6 - 64$.
20. $x^{12} + y^{12}$.
21. $27x^{27} - 1$.
22. $k^{12} - y^6$.
23. $x^{3a} + 1$.
24. $x^{3a} - y^{3a}$.
25. $x^{3a} + y^{6a}$.
26. $8x^{6a} - y^{9a}$.
27. $27x^{3m} - 64$.

REVIEW EXERCISES

Factor:

1. $x^6 + 2x^3 + 1$.
2. $x^6 - 2x^3y^3 + y^6$.
3. $x^4 + x$.
4. $x^5 - x^2$.
5. $8x^3 - 64$.
6. $8x^6 + 216x^3$.
7. $x^5 - x^3y^2 - x^2y^3 + y^5$.
8. $x^5 + x^3 + 8x^2 + 8$.
9. $x^7 - x^4 - 16x^3 + 16$.
10. $x^6 - 7x^3 - 8$.
11. $8x^6 - 3 - 23x^3$.
12. $x^4 + x^3 - x - 1$.
13. $x^6 - x^2y^2$.
14. $x^8 + 64x^2$.
15. $1 + 3x + 3x^2 + x^3$.

Solution:
$$\begin{aligned} 1 + 3x + 3x^2 + x^3 &= (1 + x^3) + (3x + 3x^2) \\ &= (1 + x)(1 - x + x^2) + 3x(1 + x) \\ &= (1 + x)(1 - x + x^2 + 3x) \\ &= (1 + x)(1 + 2x + x^2) \\ &= (1 + x)(1 + x)(1 + x). \end{aligned}$$

16. $1 - 3x + 3x^2 - x^3$.
17. $x^3 + 3x^2y + 3xy^2 + y^3$.
18. $x^3 - 3x^2y + 3xy^2 - y^3$.
19. $x^3 + 2x^2 + 4x + 8$.
20. $8 - 4x + 2x^2 - x^3$.
21. $x^3 + 9x^2 + 27x + 27$.
22. $x^3 - 9x^2 + 27x - 27$.
23. $x^3 + 12x^2 + 48x + 64$.
24. $64 - 48x + 12x^2 - x^3$.
25. $8x^3 - 12x^2 + 6x - 1$.
26. $x^4 + x^3 + x^2 - 1$.
27. $x^3 + a^3 + x^2 - a^2$.

56. General directions for factoring. Since the various methods of factoring cannot be stated in a few simple rules, they must be learned by means of such type forms and typical solutions as are given in the preceding pages. When once these have been thoroughly mastered, readiness in factoring becomes a matter of experience. Usually a student finds it comparatively easy to factor a list of exercises classified under a particular type form, yet a list of miscellaneous exercises he finds difficult. This often indicates inability to determine the type of an expression by its *form*. Until the student, by careful study of the type forms, has acquired the ability to do this, he will make little progress.

The following suggestions will prove helpful:

I. *First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.*

II. *Then by the form of the polynomial factor determine with which of the following types it should be classed and use the methods of factoring applicable to that type.*

1. $ax + ay + bx + by$.
2. $a^2 \pm 2ab + b^2$.
3. $x^2 + bx + c$.
4. $ax^2 + bx + c$.
5. $\begin{cases} a^2 - b^2. \\ a^2 + 2ab + b^2 - c^2. \\ a^2 + 2ab + b^2 - c^2 - 2cd - d^2. \end{cases}$
6. $a^3 \pm b^3$.

III. *Proceed again as in II with each polynomial factor obtained until the original expression has been separated into its prime factors.*

MISCELLANEOUS EXERCISES

Factor:

1. $x^4 + 64y^4 + 16x^2y^2$.
2. $4a^2x^4 + 4a^2x^2 - 120a^2$.
3. $9x^2 + 27x + 14$.
4. $3x^4 + x^2 - 2$.
5. $3x^3 - 75x$.
6. $x^2 + 2a^2x^2 + x^2a^4$.
7. $x^3 - 27$.
8. $x^2 - xz + 2xy - 2yz$.
9. $2ac - bc + 6ad - 3bd$.
10. $25x^2y^2 + 16x^4 - 40x^3y$.
11. $x^4 - 15x^3 + 56x^2$.
12. $9x^6 - 6x^3 - 35$.
13. $a^6 - 4096$.
14. $a^{2x+2} - 1$.
15. $4x^4 - 13x^2y^2 + 9y^4$.
16. $216 + x^3$.
17. $x^9 - 512$.
18. $x^2 - 4y^2 + 9 - 6x$.
19. $ah - 2ak + 2bh - 4bk$.
20. $4x^4 - 25y^8$.
21. $a^2 - 25c^2 + 10ab + 25b^2$.
22. $y^6 - 343$.

23. $9x^4 - 9x^2 - 28$.

24. $9x^2y^2 + 39xy^2 - 30y^2$.

25. $3x^6y^2 + 63x^3y^2 - 300y^2$.

26. $12 - 15a + 16x - 20ax$.

27. $2cx + 3dx - 2cy - 3dy$.

28. $y^4 + z^8 - 2y^2z^4$.

29. $81c^{10} - 64d^{10}$.

30. $100x^8 - 220x^4y^2 + 121y^4$.

31. $27y^3 - 512$.

32. $81 - 16x^8$.

33. $25x^4 - 3025x^8$.

34. $16c^7 - 4c^5 - 72c^3$.

35. $c^3 + c^2d + 3cd^2 + 3d^3$.

36. $1 + 81x^8 - 18x^4$.

37. $x^4 + 12x^2 - 64$.

38. $c^9 - 512c^{12}$.

39. $h^{6a} - k^{8a}$.

40. $a^2 - b^4 - 12ab + 36b^2$.

41. $5h^2 + 21hk - 20k^2$.

42. $x^6 - x^3 - 2$.

43. $2x^2y^2 - y^4 - x^4$.

44. $5h^3 - 9h^2k - 9k^3 + 5hk^2$.

45. $3x^2 - 17x + 10$.

46. $9x^{5n} - x^{3n}$.

47. $27a - 18ab^2 - 3a^9 + 3ab^4$.

48. $1 + 2x^2 - 3x^4$.

49. $x^4 - x^2 - 12$.

50. $16x^4 + 8x^2 - 3$.

51. $1024 - 64h^3 + h^6$.

52. $h^2k^2 - h^2 - k^2 + 1$.

53. $12 - 2h - 4h^2$.

54. $3 - y^2 + 3y^3 - y^5$.

CHAPTER XIV

SOLUTION OF EQUATIONS BY FACTORING

57. Definitions. A term is **rational** if it may be obtained from 1 and the letters involved by means of the four fundamental operations without the extraction of any root.

Thus $15x$, $\frac{a-b}{2}$, $\frac{8y}{3}$, and $\sqrt{4a^2}$ are rational, while $\sqrt{2x}$, $\sqrt{a-b}$, and $\sqrt{x^3}$ are irrational.

An algebraic expression is rational if its terms are rational.

An irrational term or an irrational expression may be rational with respect to a certain letter or letters.

Thus $4x^2 - \sqrt{a}\,x + 2a$ is rational with respect to x. The equation $4x^2 - \sqrt{a}\,x + 2a = 0$ is an equation rational in x.

A term is **integral** if it has no literal denominator and the exponents of the letters are *positive integers.*

Thus $\frac{3x^4}{5}$ and $4ax^2$ are integral terms. (It will be seen later what x^{-2} and $x^{\frac{2}{3}}$ mean, and why they are not integral terms.)

An expression is integral if its terms are integral.

A nonintegral term or expression may be integral with respect to a certain letter, if that letter does not occur in any denominator.

Thus the left member of $x^3 - \frac{3x}{a} - \frac{b}{a^2} = 0$ is integral with respect to x and the equation itself is integral in x.

The *degree* of a rational integral equation in one unknown is the same as the highest power of the unknown in it.

An equation of the second degree is called a **quadratic** equation.

For example, $x^2 - 6x = 18$ and $ax^2 + bx + c = 0$ are quadratic equations.

An equation of the third degree is called a **cubic.**

For example, $x^3 = 1$, $x^3 - 5x^2 + 6x + 2 = 0$, and $ax^3 + bx^2 + cx + d = 0$ are cubic equations.

An equation of the fourth degree is called a **biquadratic.**

Thus $x^4 = 16$, $x^4 + 3x^2 = 4$, and $ax^4 + bx^3 + cx^2 + dx + e = 0$ are biquadratic equations.

One important application of factoring is determining the roots of equations of the second or a higher degree.

In the solution of equations by factoring use is made of the following

PRINCIPLE. *If the product of two or more factors is zero, one of the factors must be zero.*

Two or more, or even all of the factors *may* be zero, but the vanishing of one is *sufficient* to make a product zero.

EXAMPLES

1. Solve the quadratic equation $x^2 + 5x = 6$.

Solution: Transposing, $x^2 + 5x - 6 = 0$.

Factoring, $(x-1)(x+6) = 0$.

Suppose the first factor, $x-1$, has the value zero. Then its product with the second factor is zero, no matter what value $x+6$ may have. Hence the value of x which makes $x-1$ equal to zero is a root of the quadratic.

Setting $x - 1 = 0$, we obtain $x = 1$.

Then
$$\begin{aligned}(x-1)(x+6) &= (1-1)(1+6)\\ &= (0)\cdot(7)\\ &= 0.\end{aligned}$$

Similarly if the second factor, $x+6$, has the value zero, its product with the first factor is zero, no matter what value $x-1$ may have.

Setting $x + 6 = 0$, we obtain $x = -6$.

Then
$$\begin{aligned}(x-1)(x+6) &= (-6-1)(-6+6)\\ &= (-7)\cdot(0)\\ &= 0.\end{aligned}$$

Check: Substituting 1 for x in $x^2 + 5x = 6$, $1 + 5 = 6$.

Substituting -6 for x in $x^2 + 5x = 6$, $36 - 30 = 6$.

2. Solve the cubic equation $x^3 + x^2 = 4x + 4$.

Solution: Transposing, $x^3 + x^2 - 4x - 4 = 0$.

Grouping,
$$(x^3 + x^2) + (-4x - 4) = 0.$$
$$x^2(x + 1) - 4(x + 1) = 0.$$
$$(x + 1)(x^2 - 4) = 0.$$
$$(x + 1)(x + 2)(x - 2) = 0.$$

Setting each factor equal to zero,

$x + 2 = 0$, whence $x = -2$;
$x - 2 = 0$, whence $x = +2$;
$x + 1 = 0$, whence $x = -1$.

Therefore -2, $+2$, and -1 are the roots of the equation $x^3 + x^2 = 4x + 4$.

Check:
$$\begin{cases} \text{When } x = -2, & -8 + 4 = -8 + 4. \\ \text{When } x = 2, & 8 + 4 = 8 + 4. \\ \text{When } x = -1, & -1 + 1 = -4 + 4. \end{cases}$$

For solving an equation in one unknown by factoring we have the

Rule. *Transpose the terms so that the right member is zero. Then factor the expression on the left, set each factor which contains an unknown equal to zero, and solve the resulting equations.*

It must be kept in mind that a *root* of an equation is a number which satisfies the equation.

Thus the equation $x^2 + 3x + 2 = 0$ has the two roots -2 and -1, since each of these numbers, if put for x, reduces the equation to an identity.

The two preceding examples indicate that in solving equations by factoring, two factors will arise for a quadratic, three for a cubic, four for a biquadratic, etc. Further, if the factors of a given equation are unlike, each will yield a different root. Some of the factors, however, may be alike. Thus $x^2 - 6x + 9 = 0$ can be written $(x - 3)(x - 3) = 0$. Here each factor gives the same root, 3, and though this equation is of the *second* degree, it has only one number, 3, for a root. Therefore, *an equation usually has the same number of distinct roots as the number representing its degree, but never more than that number.*

One should never divide each member of an equation by an expression containing the unknown, for in this manner roots may be lost.

EXERCISES

Find the roots of the following quadratic equations:

1. $x^2 - 9 = 0.$
2. $x^2 = 16.$
3. $x^2 - 3x = 0.$
4. $x^2 = 7x.$
5. $x^2 - 7x = -12.$
6. $x^2 - x = 20.$
7. $3x^2 - 15x = 0.$
8. $5x^2 + 35x = 0.$
9. $8 - 9x = -x^2.$
10. $12x - 28 = -x^2.$
11. $4x^2 = 16x.$
12. $x^2 - a^2 = 0.$
13. $x^2 = 9b^2.$
14. $x^2 - 2ax + a^2 = 0.$
15. $x^2 + 4b^2 = 4bx.$
16. $x^2 + ax + 3x + 3a = 0.$
17. $x^2 + bx = 4x + 4b.$
18. $x^2 + ax + bx + ab = 0.$
19. $4x^2 + 8x + 3 = 0.$
20. $9x^2 = 3x + 2.$
21. $16x^2 - 12x = 10.$
22. $50x + 24 = 25x^2.$
23. $x^2 + bx = 0.$
24. $x^2 - ax - bx = 0.$
25. $x^2 + 3x = ax.$
26. $4x = x^2 + 4.$

Solve the following cubics:

27. $x^3 - 9x = 0.$
28. $x^3 + x^2 = 4x + 4.$
29. $x^3 - 5x^2 + 6x = 0.$
30. $2x^3 - x^2 = 32x - 16.$
31. $5x^2 + x^3 = 45 + 9x.$

Find the roots of the following biquadratics:

32. $x^4 - 5x^2 + 4 = 0.$
33. $9 + x^4 = 10x^2.$
34. $x^4 - 36x^2 = 0.$
35. $x^4 + 15x^2 = 8x^3.$
36. $x^4 - 2x^2 + 1 = 0.$

37. Point out the error in the following:

Let $x = 1.$
Then $x^2 = x.$
Subtracting 1, $x^2 - 1 = x - 1.$
Factoring, $(x - 1)(x + 1) = x - 1.$
Dividing by $x - 1$, $x + 1 = 1.$
Therefore $1 + 1 = 1,$
or $2 = 1.$

PROBLEMS

1. The square of a certain number plus the number itself is 90. Find the number.

2. If from the square of a certain number twice the number be taken, the remainder will be 35. Find the number.

3. If to the square of a certain number the sum of twice the number and 5 be added, the result will be 148. Find the number.

4. Four times the square of a certain number is equal to seven times the number. What is the number?

5. A certain number is added to 20, and the same number is also added to 21; the product of the two sums is 930. What is the number?

6. A certain number is subtracted from 17, and the same number is also subtracted from 23; the product of the remainders is 216. Find the number.

7. From 27 a certain number is subtracted, and the same number is added to 21; the product of the results thus obtained is 540. Find the number.

8. If a certain number be added to 15, and the same number be subtracted from 22, the product of the sum and difference thus obtained will be 70 more than 23 times the number. Find the number.

9. The difference of two numbers is 6, and the difference of their squares is 120. Find the numbers.

10. If from the square of three times a certain number, five times the number be taken, the result will be eight times the square of the number. Find the number.

11. The depth of a certain lot whose area is 2500 square feet is four times its frontage. Find its dimensions.

12. The area of the floor of a certain room is 24 square yards. The length is 6 feet more than the breadth. What are the dimensions of the floor?

13. The area of a rectangular field is 216 square rods. The field is 6 rods longer than it is wide. Find its dimensions.

14. The sum of the squares of two consecutive numbers is 145. Find the numbers.

15. The sum of the squares of two consecutive odd numbers is 290. Find the numbers.

16. The sum of the squares of three consecutive odd numbers is 251. Find the numbers.

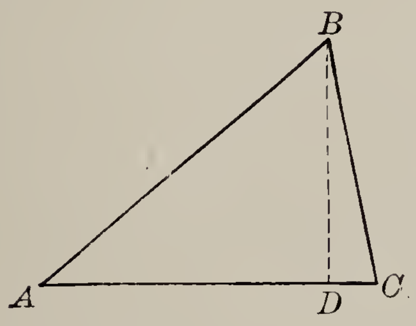
17. An uncovered square box 8 inches deep has 185 square inches of inside surface. Find the other inside dimensions.

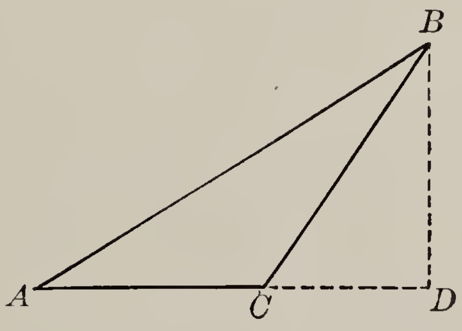
18. Remembering that the faces of a cube are squares, find the edge of a cubical box whose entire outer surface is 294 square inches.

19. A rectangular box is four times as long and three times as wide as it is deep. There are 608 square feet in its entire outer surface. Find its dimensions.

20. A box is 3 inches longer and 1 inch wider than it is deep. There are 62 square inches in its entire outer surface. Find its dimensions.

The *altitude* of a triangle is the length of a perpendicular from any vertex to the side opposite. This side is called the *base.*





In the adjacent figures BD is the altitude and AC is the base of each triangle.

If a is the altitude of a triangle and b its base, the area of the triangle is $\frac{ab}{2}$.

21. The area of a triangle is 30 square feet; its altitude is 6 feet. Find the base.

22. The altitude of a triangle is three times the base and the area is 54 square feet. Find the base and the altitude.

23. The base of a triangle is five times the altitude and the area is 40 square feet. Find the base and the altitude.

24. The area of a triangle is 75 square meters; the base is six times the altitude. Find the altitude and the base.

25. The area of a triangle is 24 square feet; the altitude is 2 feet longer than the base. Find the altitude and the base.

HINT. Let $x =$ the base in feet.

Then $x + 2 =$ the altitude in feet,

and $\frac{x(x+2)}{2}$, or $\frac{x^2 + 2x}{2} =$ the area.

Therefore $$\frac{x^2 + 2x}{2} = 24.$$

Multiplying each member by 2, this equation becomes

$$x^2 + 2x = 48.$$

26. The altitude of a triangle is 3 feet longer than the base and the area is 6 square yards. Find the base and the altitude.

27. One leg of a right triangle is 2 feet longer than the other and the area is 24 square feet. Find the legs.

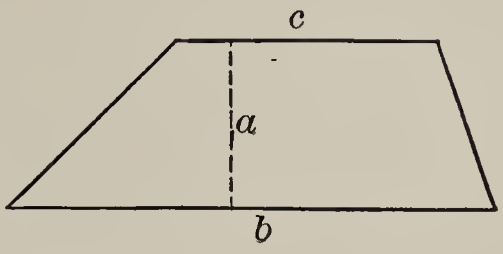
28. The area of a right triangle is 30 square feet and one leg is 7 feet longer than the other. Find the legs.

29. The area of a triangle is $2\frac{5}{8}$ square feet and the base is 6 inches longer than twice the altitude. Find the base and altitude.

30. The area of a triangle is 4 square yards and the altitude is 6 feet more than three times the base. Find the base and the altitude.

31. The area of a triangle is .015 square meters. The altitude is 5 centimeters shorter than the base. Find the dimensions.

A *trapezoid* is a four-sided figure, two of whose sides are unequal and parallel.



The bases of the adjacent trapezoid are the two parallel sides b and c.

The altitude, a, of the trapezoid is the perpendicular distance between the bases.

The area of a trapezoid is $\dfrac{a(b+c)}{2}$.

32. Find the area of a trapezoid whose bases are 10 and 18 and whose altitude is 12.

33. The altitude of a trapezoid is 8 inches, its area is 96 square inches, and one base is 4 inches longer than the other. Find each base.

HINT. Let $x =$ the length of one base in inches.
Then $x + 4 =$ the length of the other base in inches,
and $\dfrac{8(x+x+4)}{2}$, or $8x+16 =$ area of the trapezoid.
Therefore $8x + 16 = 96.$

34. One base of a trapezoid is 12 feet, the other base is twice the altitude, and the area is 112 square feet. Find the altitude.

35. The altitude of a trapezoid is $\frac{1}{2}$ the shorter base and the latter is $\frac{2}{3}$ of the other base. The area is 360 square feet. Find the bases and the altitude.

36. One base of a trapezoid is 4 feet longer than the altitude, the other base is 6 feet longer than the altitude, and the area is 66 square feet. Find the bases and the altitude.

37. The bases of a trapezoid are respectively 8 feet and 12 feet longer than the altitude, and the area is 16 square yards. Find the bases and the altitude.

38. One base of a trapezoid is 4 feet longer than the other, the altitude is $\frac{1}{2}$ the sum of the bases, and the area is 4 square yards. Find the bases and the altitude.

39. The area of a trapezoid is 10 square yards, the altitude equals one base, and the other base exceeds the altitude by 2 feet. Find the bases and the altitude.

40. One base of a trapezoid exceeds the other by 10 feet, the altitude is 2 feet longer than five times the shorter base, and the area is 22 square yards. Find the altitude and the two bases.

41. The area of a trapezoid is .09 square meters. One base is twice the other and the altitude 10 centimeters less than the longer base. Find the bases and the altitude.

CHAPTER XV

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

58. Highest common factor. The degree of a rational, integral monomial is determined by the sum of the exponents of the letters in it.

Thus ax^3 is of the fourth degree, and $4\,a^3x^3y^2$ is of the eighth degree.

The degree of a rational, integral polynomial is the same as that of its term of highest degree.

Thus $6\,axy^2 + 4\,a^3x^4 - 3\,ax^2yz$ is of the seventh degree.

The **highest common factor** (H.C.F.) of two or more rational, integral expressions is the rational, integral expression of highest degree with the greatest numerical coefficient which is an exact divisor of each.

Thus the H.C.F. of $36\,a^2b^2$ and $48\,a^3b$ is $12\,a^2b$. The H.C.F. of $x^3 - 4\,x$ and $x^3 - 5\,x^2 + 6\,x$ is $x^2 - 2\,x$.

The problem of finding the H.C.F. of polynomials which cannot be factored by inspection will not be considered here, as it is not necessary to find the H.C.F. of such expressions in elementary work.

EXAMPLES

1. Find the H.C.F. of $24\,x^3y^2z$, $48\,x^4y^6z^3$, and $72\,x^5y^5z^2$.

Solution: Factoring, we have

$$24\,x^3y^2z = 2^3 \cdot 3\,x^3y^2z,$$
$$48\,x^4y^6z^3 = 2^4 \cdot 3\,x^4y^6z^3,$$
$$72\,x^5y^5z^2 = 2^3 \cdot 3^2x^5y^5z^2.$$

Here the highest power of 2 common to each expression is the third, of 3 the first, of x the third, of y the second, and of z the first. Therefore the H.C.F. of the three expressions is $2^3 \cdot 3 \cdot x^3y^2z$, which equals $24\,x^3y^2z$.

2. Find the H.C.F. of $2x^4 - 12x^3 + 18x^2$ and $4x^5 - 36x^3$.

Solution: Factoring, we have

$$2x^4 - 12x^3 + 18x^2 = 2x^2(x-3)^2,$$
$$4x^5 - 36x^3 = 2^2x^3(x-3)(x+3).$$

Therefore the H.C.F. is $2x^2(x-3)$, which equals $2x^3 - 6x^2$.

The method of the preceding solutions for finding the H.C.F. of two or more rational, integral expressions is stated in the

Rule. *Separate each expression into its prime factors. Then find the product of such factors as occur in each expression, using each the least number of times it occurs in any one expression.*

EXERCISES

Find the H.C.F. of the following:

1. 12, 18.
2. 24, 56.
3. 96, 144.
4. 84, 196.
5. 125, 225.
6. 64, 96, 256.
7. 90, 108, 324.
8. $12x^4$ and $18x^3$.
9. $16x^2y^4$ and $24x^3y^5$.
10. $27c^2d$ and $21cd^4$.
11. $32a^3bc^4$ and $48a^2b^2d$.
12. $125m^7n^5p$ and $100nq^4$.
13. $18h^2k^2$, $36h^4k$, and $24h^3k^3$.
14. $9xy^4$, $54x^5y$, and $15x^4y^5$.
15. $27a^4b^5c^2$, $54a^3b^2d$, and $81a^2b^2c^3$.
16. $x^2 - 9$ and $x^2 - 5x + 6$.
17. $x^2 + 3x - 10$ and $x^2 + 6x + 5$.
18. $x^3 - 4x$ and $x^3 - 8x^2 + 12x$.
19. $2c^3 + 12c^2 + 18c$ and $c^3 - 2c^2 - 15c$.
20. $8 + y^3$ and $y^2 + 4y + 4$.
21. $x^4 - 2x^2 + 1$ and $x^2 - 2x + 1$.
22. $ab + 3b + ac + 3c$ and $2ab + 6b - 2ac - 6c$.
23. $c^2 + 3cd + 2d^2$, $c^2 + 5cd + 6d^2$, and $c^2 + cd - 2d^2$.

59. Lowest common multiple. The **lowest common multiple** (L.C.M.) of two or more rational, integral expressions is the rational, integral expression of lowest degree which will exactly contain each.

EXAMPLES

1. Find the L.C.M. of $24\,x^3y^2$, $36\,x^4y$, and $54\,x^2y^2z$.

Solution:

$$24\,x^3y^2 = 2^3\cdot 3\cdot x^3y^2,$$
$$36\,x^4y = 2^2\cdot 3^2\cdot x^4y,$$
$$54\,x^2y^2z = 2^1\cdot 3^3\cdot x^2y^2z.$$

Since the L.C.M. must contain each of the expressions, it must have 2^3 as a factor. If the L.C.M. has 2^3 as a factor, it will contain 2^2 and 2^1 which occur in the second and third monomials respectively. Similarly, the L.C.M. must contain as factors 3^3, x^4, y^2, and z. Therefore the L.C.M. is $2^3\cdot 3^3\cdot x^4y^2z$, which equals $216\,x^4y^2z$.

2. Find the L.C.M. of $9\,a^2y + 18\,aby + 9\,b^2y$, $15\,ax^2 - 15\,bx^2$, and $2\,a^2 - 4\,ab + 2\,b^2$.

Solution:

$$9\,a^2y + 18\,aby + 9\,b^2y = 3^2y(a+b)^2,$$
$$15\,ax^2 - 15\,bx^2 = 3\cdot 5\,x^2(a-b),$$
$$2\,a^2 - 4\,ab + 2\,b^2 = 2\,(a-b)^2.$$

In order that the L.C.M. may exactly contain each of the three expressions, it must have 2^1, 3^2, 5^1, x^2, y, $(a+b)^2$, and $(a-b)^2$ as factors. Hence the L.C.M. is $2\cdot 3^2\cdot 5^1\,x^2y(a+b)^2(a-b)^2$, which equals $90\,a^4x^2y - 180\,a^2b^2x^2y + 90\,b^4x^2y$.

The method of finding the L.C.M. of two or more rational, integral expressions is stated in the following

RULE. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

EXERCISES

Find the L.C.M. of:

1. 12, 18.

2. 32, 48.

3. 20, 28.

4. 96, 144.

5. 64, 120, 216.

6. 128, 160, 200.

7. x^2y, xy^2, xy^3.

8. $6\,cd^3$, $4\,c^5de$, $10c^2d^2e^3$.

9. $8\,abc$, $3\,b^2c$, $12\,c^2$.

10. $18\,m^2$, $15\,mn^2p$, $20\,m^3p^2$.

11. 20, $18\,xy^4$, $27\,ax^3y$.

12. $36\,ab^2$, $42\,abc$, $63\,b^2c$.

13. $4\,a$, $a^2 - ab$.

14. $12\,ax$, $3\,a^3x^2 - 3\,ax^3$.

15. $cx + cy$, $dy + dx$.

16. $3\,x + 3\,z$, $6\,a^2x + 6\,a^2z$.

17. $x^2 - xy$, $ax + ay$.

18. $x^2 - 9$, $x^2 - 5x + 6$.

19. $c^2 - 4$, $c^2 - 8c - 20$.

20. $4ax$, $4x^2 - 1$, and $4x^2 + 4x + 1$.

21. $x^2 + 1$, $x^4 - 1$, and $x^4 - 2x^2 + 1$.

22. $4 - c^2$, $c^3 + 8$, and $c^2 + 6c + 8$.

23. $ac - 2bd + 2ad - bc$, $a^2 + ab - 2b^2$.

24. $x^2 - y^2$, $x^3 - y^3$, and $x^2 + 2xy + y^2$.

25. $ax^2 + bxy$, $2ax + 2by$, and $a^2x^2 - b^2y^2$.

26. $2x^3 - 2x$, $3x^4 + 15x^3 - 18x^2$, and $x^2 - 36$.

27. $8 - y^3$, $y^2 - 4$, and $4y^2 + 2y^3 + y^4$.

28. $2x^3 - 6x^2 + 4x^2y$ and $x^2 - 4y^2 - 6x + 9$.

CHAPTER XVI

FRACTIONS

60. Algebraic fractions. The expression $\frac{a}{b}$, in which a and b represent numbers or polynomials, is an **algebraic** fraction. It is read "a divided by b," or "a over b." A fraction is an indicated quotient in which the dividend is called the numerator and the divisor the denominator. The numerator and denominator are often called the *terms* of a fraction.

As division by zero has no meaning, **the denominator of a fraction can never be zero.**

The reduction of a fraction to lower or to higher terms, the addition of fractions, and the subtraction of fractions in both arithmetic and algebra depend on the

PRINCIPLE. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus $$\frac{3}{4}=\frac{3\cdot4}{4\cdot4}=\frac{12}{16}, \text{ and } \frac{18}{30}=\frac{18\div6}{30\div6}=\frac{3}{5}.$$

Similarly $$\frac{a}{b}=\frac{a\cdot n}{b\cdot n}=\frac{an}{bn}, \text{ and } \frac{a}{b}=\frac{a\div n}{b\div n}=\frac{a/n}{b/n}.$$

Since $\frac{4}{4}$, $\frac{6}{6}$, and $\frac{n}{n}$ are each equal to 1, each of the four preceding illustrations is really a multiplication or a division of a fraction by 1. This produces no change in the numerical value of *any* fraction, though it may change its form.

61. Reduction of fractions to lowest terms. A rational fraction is in its **lowest terms** when no rational factor except 1 is common to both numerator and denominator.

Cancellation is the process of dividing the numerator and the denominator of a fraction by a factor common to both.

EXAMPLES

Reduce to lowest terms:

1. $\dfrac{72\,a^3x^2y^5}{108\,a^2x^4y^2}$.

$$\textbf{Solution:}\quad \frac{72\,a^3x^2y^5}{108\,a^2x^4y^2}=\frac{\overset{2}{\cancel{2^3}}\cdot\cancel{3^2}\cdot\overset{a}{\cancel{a^3}}\cdot\cancel{x^2}\cdot\overset{y^3}{\cancel{y^5}}}{\cancel{2^2}\cdot\underset{3}{\cancel{3^3}}\cdot\cancel{a^2}\cdot\underset{x^2}{\cancel{x^4}}\cdot\cancel{y^2}}=\frac{2\,ay^3}{3\,x^2}.$$

2. $\dfrac{2\,c^5-32\,c}{4\,c^6+16\,c^4-128\,c^2}$.

$$\textbf{Solution:}\quad \frac{2\,c^5-32\,c}{4\,c^6+16\,c^4-128\,c^2}=\frac{\cancel{2c}(c^2+4)\cancel{(c+2)}\cancel{(c-2)}}{\underset{2\,c}{\cancel{4c^2}}(c^2+8)\cancel{(c+2)}\cancel{(c-2)}}=\frac{c^2+4}{2\,c^3+16\,c}.$$

For reducing a fraction to its lowest terms the preceding examples illustrate the

RULE. *Separate the numerator and the denominator into their prime factors and cancel the factors common to both.*

Cancellation as used in the rule means an actual division of the numerator and the denominator by the same expression. Therefore *only* **factors** *which are common to the numerator and the denominator can be cancelled.*

The terms (the parts connected by plus or minus signs) in polynomial numerators and denominators, even if alike, can never be cancelled. For example, $\frac{5+2}{6+2}=\frac{7}{8}$. Here it would be incorrect to "cancel" thus: $\frac{5+\cancel{2}}{6+\cancel{2}}$, as the resulting fraction would be $\frac{5}{6}$. Similarly, in the fraction $\frac{x+a+4\,c^2}{y+a+8\,c^2}$, no cancellation is possible.

We have seen that we may multiply or divide both numerator and denominator of a fraction by the same number without affecting the value of the fraction. But we should never forget that adding the same number to or subtracting the same number from both numerator and denominator changes the value of the fraction. Also squaring both numerator and denominator leads to a different value. Compare this statement with the operations that may be performed on each member of an equation as given on page 33.

EXERCISES

Reduce to lowest terms :

1. $\frac{a^3b}{a^2b^4}$.
2. $\frac{12\,x^2y^6}{18\,xy^7}$.
3. $\frac{32\,a^4b^4c}{48\,a^3b^6c^2}$.
4. $\frac{45\,cd^5e^2}{20\,c^5d^3}$.
5. $\frac{36\,a^2b^2}{54\,a^5bc^3}$.
6. $\frac{2\,a}{2\,a+2}$.
7. $\frac{3\,xy+3\,y^2}{3\,y^4}$.
8. $\frac{21\,d^2+14\,c}{14\,c}$.
9. $\frac{a^2-1}{a^2-2\,a+1}$.
10. $\frac{4\,x^2+4\,x+1}{4\,x^2-1}$.
11. $\frac{2\,a^2-2\,b^2}{4\,a^2-8\,ab+4\,b^2}$.
12. $\frac{18\,ax^3+9\,ay}{12\,bx^3+6\,by}$.
13. $\frac{x^2-25}{x^2-x-30}$.
14. $\frac{c^2-5\,cd+4\,d^2}{c^2-16\,d^2}$.
15. $\frac{2\,x^2-2\,x-180}{2\,x^2-162}$.
16. $\frac{21+10\,x+x^2}{x^2-9}$.
17. $\frac{4\,c^3-5\,c^2-4\,c+5}{8\,c^4-10\,c^3+12\,c-15}$.
18. $\frac{x^2-y^2}{(x-y)^2}$.
19. $\frac{a^2-b^2}{(a-b)^3}$.
20. $\frac{c^2-d^2}{c^3-d^3}$.
21. $\frac{y^3-z^3}{(y-z)^3}$.
22. $\frac{x^3-8}{x^2-4}$.
23. $\frac{x^6-1}{x^2-1}$.
24. $\frac{x^4-y^4}{x^6-y^6}$.
25. $\frac{64\,x^3+1}{1+8\,x+16\,x^2}$.

62. Changes of sign in a fraction. The *sign of a fraction* is the plus or minus sign placed before the line separating the numerator from the denominator. Hence there are in a fraction three signs to consider, — the sign of the fraction, the sign of the numerator, and the sign of the denominator.

Now in division the quotient of two expressions having like signs is positive, and the quotient of two expressions having unlike signs is negative.

Therefore $+\frac{+8}{+2}=+4;\qquad +\frac{-8}{-2}=+4;$

$$-\frac{-8}{+2}=-(-4)=+4;\quad -\frac{+8}{-2}=-(-4)=+4.$$

Or, in general terms, $+\frac{+a}{+b}=+\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}.$

These examples illustrate the

Principle. *In a fraction the signs of both numerator and denominator, or the sign of the numerator and the sign before the fraction, or the sign of the denominator and the sign before the fraction may be changed without altering the value of the fraction.*

Hence any fraction may be written in at least four ways, if proper changes of sign are made.

Thus $\quad +\frac{2a}{x-3}=+\frac{-2a}{-x+3}=-\frac{-2a}{x-3}=-\frac{+2a}{-x+3}.$

Similarly

$$+\frac{2x-5}{x-2y+4}=+\frac{-2x+5}{-x+2y-4}=-\frac{-2x+5}{x-2y+4}=-\frac{2x-5}{-x+2y-4}.$$

EXERCISES

Write in three other ways each of the following:

1. $\frac{-x}{y}.$
2. $\frac{x}{-y}.$
3. $\frac{a}{a-b}.$
4. $\frac{x-3}{2x+4}.$
5. $-\frac{c^2-d}{-2c+d}.$
6. $-\frac{x-y}{x+y-3}.$
7. $\frac{x^2-5x+6}{x^2-7x+12}.$
8. $-\frac{-x+x^2-2}{x^3-x^2y-1}.$

63. Equivalent fractions. Two fractions are **equivalent** when one can be obtained from the other by multiplying or by dividing both of its terms by the same expression.

For example, $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions; also $\frac{a^2}{ab}$ and $\frac{a}{b}.$

The lowest common denominator (L.C.D.) of two or more fractions is the L.C.M. of their denominators.

EXAMPLES

Reduce to equivalent fractions having the lowest common denominator:

1. $\frac{5a}{6b^2c}$ and $\frac{3b}{4ac^2}$.

Solution: The L.C.M. of the denominators is $12ab^2c^2$. Multiplying both numerator and denominator of the first fraction by the factor $2ac$, which is found in the L.C.M. but not in the denominator of the fraction, gives $\frac{10a^2c}{12ab^2c^2}$. Multiplying both numerator and denominator of the second fraction by the factor $3b^2$, which is found in the L.C.M. but not in the denominator of this fraction, gives $\frac{9b^3}{12ab^2c^2}$. Hence the required fractions are $\frac{10a^2c}{12ab^2c^2}$ and $\frac{9b^3}{12ab^2c^2}$.

2. $\frac{3x-1}{2x^3-18x}$ and $\frac{5}{x^2-5x+6}$.

Solution: Factoring the denominators and rewriting gives

$$\frac{3x-1}{2x(x+3)(x-3)} \text{ and } \frac{5}{(x-3)(x-2)}.$$

By inspection the L.C.D. is seen to be $2x(x+3)(x-3)(x-2)$. Multiplying both terms of the first fraction by the factor $x-2$, which is found in the L.C.D. but not in the denominator of the fraction, gives $\frac{(3x-1)(x-2)}{2x(x+3)(x-3)(x-2)}$, or $\frac{3x^2-7x+2}{2x^4-4x^3-18x^2+36x}$. Multiplying both terms of the second fraction by $2x(x+3)$, found in the L.C.D. but not in the denominator of the fraction, gives $\frac{5\cdot 2x(x+3)}{(x-3)(x-2)2x(x+3)}$, or $\frac{10x^2+30x}{2x^4-4x^3-18x^2+36x}$.

Therefore, to change two or more fractions (in their lowest terms) to equivalent fractions having the L.C.D., we have the

RULE. *Rewrite the fractions with their denominators in factored form.*

Find the L.C.M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L.C.M. which are not found in the denominator of the fraction.

EXERCISES

Change the following fractions to equivalent fractions having the lowest common denominator:

1. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}.$
2. $\frac{5}{24}, \frac{7}{32}.$
3. $\frac{4}{3d}, \frac{2c}{5d}.$
4. $\frac{3a+b}{12}, \frac{3a}{16}.$
5. $\frac{5x+y}{21}, \frac{4x+3y}{35}.$
6. $\frac{4}{5a^2b}, \frac{3}{abc}.$
7. $\frac{5c}{3mn^3}, \frac{12d}{11m^2n^2}.$
8. $\frac{3a}{2b^2c}, \frac{2b}{3a^2c}$, and $\frac{c}{6ab}.$
9. $\frac{3b}{2cd^3}, \frac{9a}{4cde}$, and $\frac{7ab}{5cde^2}.$
10. $\frac{x+y}{3xy^2}, \frac{x-2y}{2x^2y}.$
11. $\frac{2+m^2}{2m}, \frac{4+m^2}{4n}.$
12. $\frac{2}{x+2}, \frac{4}{3x+6}.$
13. $\frac{4}{x+1}, \frac{5}{x-1}.$
14. $\frac{3c}{c^2-d^2}, \frac{4}{c+d}.$
15. $\frac{3x}{x+3}, \frac{5}{x^2+5x+6}.$
16. $\frac{2x}{x^2-xy}, \frac{3y}{x^2-2xy+y^2}.$
17. $\frac{3c+2}{c^2-d^2}, \frac{4-c}{c^2-7cd+6d^2}.$
18. $\frac{2x+5}{x^2-1}, \frac{2x-4}{x^3-3x^2+2x}.$

Note. The problem of operating with fractions presented great difficulties to all the early races. The Egyptians and the Greeks, even down to the sixth century of our era, always reduced their fractions to the sum of several fractions, each of which had 1 for a numerator. For instance, $\frac{5}{8}$ would be expressed as $\frac{1}{2}+\frac{1}{8}$. The Romans usually expressed all the fractions of a sum in terms of fractions with the common denominator 12. The Babylonians resorted to a similar device, but used 60 for the denominator. In some way they all attempted to evade the difficulty of considering changes in both numerator and denominator. The Hindus seem to have been the first to reduce fractions to a common denominator, though Euclid (300 B.C.) was familiar with the method of finding the least common multiple of two or more numbers.

64. Addition and subtraction of fractions. If two or more fractions have the same denominator, their sum is the fraction obtained by adding their numerators and writing the result over their common denominator.

For example, $\frac{1}{9}+\frac{2}{9}+\frac{4}{9}=\frac{7}{9}$, and $\frac{a}{b}+\frac{2a}{b}+\frac{3a}{b}=\frac{6a}{b}$.

If two fractions have the same denominator, their difference is the fraction obtained by subtracting the numerator of the subtrahend from the numerator of the minuend and writing the result over their common denominator.

For example, $\frac{5}{7}-\frac{2}{7}=\frac{3}{7}$, and $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.

If it is required to add or to subtract two fractions having unlike denominators, the fractions must be changed to equivalent fractions having a common denominator; then their sum or their difference is obtained as above.

For example, to find the sum of $\frac{1}{6}+\frac{3}{4}+\frac{2}{3}$, we reduce the fractions to equivalent fractions having a common denominator by multiplying both terms of $\frac{1}{6}$ by 2, of $\frac{3}{4}$ by 3, and of $\frac{2}{3}$ by 4. The fractions become $\frac{2}{12}$, $\frac{9}{12}$, and $\frac{8}{12}$ respectively, and their sum is $\frac{19}{12}$.

In adding unlike algebraic fractions, as $\frac{a}{b}$ and $\frac{c}{d}$, we treat them in a similar way. Multiply both terms of $\frac{a}{b}$ by d, and both terms of $\frac{c}{d}$ by b. The fractions become $\frac{ad}{bd}$ and $\frac{bc}{bd}$ respectively, whose sum is $\frac{ad+bc}{bd}$. Similarly $\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}$, which equals $\frac{ad-bc}{bd}$.

EXAMPLES

1. Simplify $\frac{6x+1}{3x^2}+\frac{3x-5y}{4xy}$.

Solution: The L.C.D. is $12x^2y$. The work of reducing the two fractions to equivalent fractions whose denominator is $12x^2y$, and of adding the resulting fractions, follows.

$$\frac{6x+1}{3x^2}+\frac{3x-5y}{4xy}=\frac{(6x+1)4y}{3x^2\cdot 4y}+\frac{(3x-5y)3x}{4xy\cdot 3x}$$

$$\frac{6x+1}{3x^2}+\frac{3x-5y}{4xy}=\frac{24xy+4y}{12x^2y}+\frac{9x^2-15xy}{12x^2y}$$
$$=\frac{24xy+4y+9x^2-15xy}{12x^2y}=\frac{9xy+4y+9x^2}{12x^2y}.$$

Check: Setting the original expression equal to the final result and substituting 1 for both x and y we obtain:

$$\frac{6x+1}{3x^2}+\frac{3x-5y}{4xy}=\frac{9xy+4y+9x^2}{12x^2y}.$$
$$\frac{6+1}{3}+\frac{3-5}{4}=\frac{9+4+9}{12}.$$
$$\frac{7}{3}+\frac{-2}{4}=\frac{22}{12},\quad\text{or}\quad\frac{22}{12}=\frac{22}{12}.$$

At this point the student should read the rule on page 143 and then solve Exercises 1–20, page 144.

2. Find the algebraic sum of $\frac{2x-3}{x^2-9}-\frac{3x-4}{x^2-5x+6}+\frac{-2}{x+3}$.

Solution: Rewriting the fractions with their denominators in the factored form, we get $\frac{2x-3}{(x+3)(x-3)}-\frac{3x-4}{(x-3)(x-2)}+\frac{-2}{x+3}$. The L.C.D. is $(x+3)(x-3)(x-2)$. The work of reducing these fractions to equivalent fractions and finding their algebraic sum follows.

$$\frac{2x-3}{(x+3)(x-3)}-\frac{3x-4}{(x-3)(x-2)}+\frac{-2}{x+3}$$
$$=\frac{(2x-3)(x-2)}{(x+3)(x-3)(x-2)}-\frac{(3x-4)(x+3)}{(x-3)(x-2)(x+3)}+\frac{-2(x-3)(x-2)}{(x+3)(x-3)(x-2)}$$
$$=\frac{2x^2-7x+6-(3x^2+5x-12)+(-2x^2+10x-12)}{(x+3)(x-3)(x-2)}$$
$$=\frac{2x^2-7x+6-3x^2-5x+12-2x^2+10x-12}{(x+3)(x-3)(x-2)}$$
$$=\frac{-3x^2-2x+6}{(x+3)(x-3)(x-2)},\quad\text{or}\quad\frac{-3x^2-2x+6}{x^3-2x^2-9x+18}.$$

Check: Proceed as in Example 1, substituting 1 for x.

$$\frac{2x-3}{x^2-9}-\frac{3x-4}{x^2-5x+6}+\frac{-2}{x+3}=\frac{-3x^2-2x+6}{x^3-2x^2-9x+18}.$$
$$\frac{2-3}{1-9}-\frac{3-4}{1-5+6}+\frac{-2}{1+3}=\frac{-3-2+6}{1-2-9+18}.$$
$$\tfrac{1}{8}+\tfrac{1}{2}-\tfrac{1}{2}=\tfrac{1}{8}.$$

In checking work in fractions, *such values must be chosen for the letters as will make no denominator zero.* This prevents the substitution of 2, 3, or -3 for x in checking the foregoing example.

3. Simplify $3x-4-\frac{x^2-5x}{x+2}$.

Solution: This may be written $\frac{3x-4}{1}-\frac{x^2-5x}{x+2}$.

The L.C.M. of the denominators is $x+2$. Multiplying both terms of the first fraction by $x+2$ and leaving the second unchanged, we get

$$\begin{aligned} 3x-4-\frac{x^2-5x}{x+2} &= \frac{(3x-4)(x+2)}{1(x+2)}-\frac{x^2-5x}{x+2} \\ &= \frac{3x^2+2x-8-(x^2-5x)}{x+2} \\ &= \frac{3x^2+2x-8-x^2+5x}{x+2} \\ &= \frac{2x^2+7x-8}{x+2}. \end{aligned}$$

Check: Let $x=2$.

$$\begin{aligned} 3x-4-\frac{x^2-5x}{x+2} &= \frac{2x^2+7x-8}{x+2}. \\ 6-4-\frac{4-10}{2+2} &= \frac{8+14-8}{2+2}. \\ 2-\frac{-6}{4} &= \frac{14}{4}. \\ \tfrac{14}{4} &= \tfrac{14}{4}. \end{aligned}$$

Therefore, to find the algebraic sum of two or more fractions (in their lowest terms), we have the

RULE. *Reduce the fractions to equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each numerator in a parenthesis preceded by the sign of the corresponding fraction.*

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator and, if necessary, reduce the resulting fraction to its lowest terms.

EXERCISES

Find the algebraic sum of:

1. $\frac{2x}{3}+\frac{3x}{5}$.

2. $\frac{2a}{3}-\frac{5a}{6}+\frac{a}{4}$.

3. $\frac{3c}{7}+\frac{c}{21}-\frac{1}{3}$.

4. $\frac{9a}{16}-\frac{12m}{8}-\frac{5m}{24}$.

5. $\frac{x-3}{4}+\frac{2x-5}{6}$.

6. $\frac{5a+7}{14}-\frac{9a+x}{21}$.

7. $\frac{5c}{3}-\frac{3c-x}{5}+\frac{x-3c}{10}$.

8. $\frac{4m-3}{6}-\frac{7-9m}{9}-\frac{3a+5m-4}{27}$.

9. $\frac{2}{a}+\frac{3}{4a^2}$.

10. $\frac{3}{x^2}+\frac{4}{x}-\frac{8}{3x^3}$.

11. $\frac{x}{y}-\frac{5}{2y^3}+\frac{a}{4}$.

12. $\frac{a}{b}+\frac{b}{c}-\frac{c}{a}$.

13. $\frac{7}{m^2}-\frac{5}{mn}+\frac{6}{n^2}$.

14. $\frac{3}{xy}-\frac{5}{2x^2}-\frac{6}{5xy^3}$.

15. $\frac{3x-1}{6x^2}-\frac{7-2x^2}{4x^3}+\frac{5x^3-8}{9x^4}$.

16. $\frac{4c^2-9}{2cd}-\frac{6-c}{5c^2}-\frac{c^2-4}{cd^2}$.

17. $\frac{3+x}{x-3}+\frac{3}{4}$.

18. $\frac{3}{x-5}-\frac{2}{x+5}$.

19. $\frac{5x}{x^2+xy}-\frac{7}{x}$.

20. $\frac{3a-b}{a^2-b^2}+\frac{2}{a-b}$.

In the solution of exercises similar to 21–30 the student should follow the method of Example 2, page 142.

21. $\frac{10}{25-m^2}-\frac{1}{m^2+16m+55}$.

22. $\frac{3}{a^2-16}+\frac{5}{a^2-6a+8}$.

23. $\frac{2x+1}{x^2-1}+\frac{4}{x^2-3x+2}$.

24. $\frac{c-5}{c^2-6c}+\frac{2c-3}{c^2-8c+12}$.

25. $\frac{x^2-3xy+y^2}{9-6x+x^2}-\frac{5x-3y}{9-3x}$.

26. $\dfrac{x+2}{x^2+x}+\dfrac{1}{x}+\dfrac{3-x}{x^2+2x+1}$.

27. $\dfrac{a+b}{a^2-ab}+\dfrac{a-2b}{a^2-2ab+b^2}-\dfrac{2a-b}{a^2-b^2}$.

28. $\dfrac{m-2n}{m^2+mn+n^2}-\dfrac{m^2-3n^2}{m^3-n^3}+\dfrac{3m-n}{m-n}$.

29. $\dfrac{c^2+cd+d^2}{c^2-cd+d^2}-\dfrac{c-d}{2c+2d}-\dfrac{2c^2+5cd}{c^3+d^3}$.

30. $\dfrac{x+y}{x^3-y^3}-\dfrac{x-2y}{(x-y)^3}$.

31. $R^2+\dfrac{R^2}{4}$.

HINT. See Example 3, page 143.

32. $x-3-\dfrac{x+3}{5}$.

33. $\dfrac{m^2+n^2}{m-n}+n+m$.

34. $3a+b-\dfrac{a^2-3b^2}{3a-b}$.

35. $x^2+y^2-\dfrac{3x^3-y^3}{x+y}-xy$.

36. $m^2-\dfrac{m^4-2m^2}{m^2+m+1}+1-m$.

37. $\dfrac{m^3+8n^3}{m^2+2mn+4n^2}+m-2n$.

38. $a^2-\dfrac{a^4+2a^2b^2-b^4}{a^2+ab+b^2}-ab+b^2$.

39. $x^3+x^2y-\dfrac{x^4+3y^4}{x-y}+xy^2+y^3$.

40. $\dfrac{2c^2+d^2}{c^2-d^2}-\dfrac{c}{c+d}+5$.

41. $6+\dfrac{6r^2-s^2}{r^2-9rs+14s^2}-\dfrac{3r-5s}{r-7s}+5$.

42. $\left(x-3-\dfrac{3}{x+3}\right)-\left(2x-7-\dfrac{4}{x+4}\right)$.

HINT. Removing parentheses, we get $x-3-\dfrac{3}{x+3}-2x+7+\dfrac{4}{x+4}$.

Combining like terms, gives $-x+4-\dfrac{3}{x+3}+\dfrac{4}{x+4}$, etc.

43. $\left(5m + \frac{6m}{2n}\right) - \left(m - \frac{m - 3n}{n}\right).$

44. $\left(7c - \frac{2d}{3de}\right) - \left(5c + \frac{c}{3de}\right).$

45. $\left(\frac{2a}{a - b} - 5a\right) - \left(2a - \frac{4a}{a + b}\right).$

46. $\left(2a - 3b + \frac{3a}{2a + b}\right) - \left(3a + 2b - \frac{5a}{2a - b}\right).$

47. $\left(- \frac{4r^2}{2r + s} + 2r - s\right) - \left(- \frac{4s^2}{2s - 3r} + 2s + 3r\right).$

48. $\frac{x - 1}{2 - x} + \frac{x^2 - 3}{x^2 - 4}.$

HINT. These fractions may be written $\frac{x - 1}{2 - x} + \frac{x^2 - 3}{(x + 2)(x - 2)}.$ Apparently the L.C.M. of the denominators is $(2 - x)(x + 2)(x - 2)$, but if both terms of the first fraction be multiplied by -1, we obtain $\frac{1 - x}{x - 2}.$ The L.C.M. of the denominators of the fractions $\frac{1 - x}{x - 2}$ and $\frac{x^2 - 3}{x^2 - 4}$ is $(x + 2)(x - 2)$.

49. $\frac{5}{x - 3} + \frac{3}{3 - x}.$

50. $\frac{6}{x^2 - 25} - \frac{3}{5 - x}.$

51. $\frac{3a}{a^2 - 4} + \frac{2a - 1}{2 - a}.$

52. $\frac{3c}{9 - c^2} - \frac{4c - 2}{c - 3}.$

53. $\frac{x}{2x - 1} - \frac{x}{1 + 2x} - \frac{x}{1 - 4x^2}.$

54. $\frac{7}{x^2 - 13x + 42} + \frac{4x - 1}{7 - x} - \frac{2x + 3}{6 - x}.$

55. $\frac{3x - 1}{x^2 + 7x - 8} - \frac{4x - 1}{1 - x} + \frac{x + 2}{8 + x}.$

Multiplying one factor of an indicated product by -1 changes the sign of every term of the expanded product.

Thus $(x - 2)(x - 3) = x^2 - 5x + 6$. Multiplying the terms of the factor $x - 2$ by -1, we have $(2 - x)(x - 3)$, or $-x^2 + 5x - 6$.

Multiplying the terms in two factors of an indicated product by -1 does not change the sign of the expanded product.

Thus $$(x-2)(x-3)=x^2-5x+6.$$
But $$\begin{aligned}(x-2)(-1)(x-3)(-1)&=(2-x)(3-x)\\&=x^2-5x+6.\end{aligned}$$

In general, changing the sign of an *odd* number of factors in an indicated product changes the sign of every term of the expanded product; but if an *even* number of factors are thus treated, the expanded product is unchanged.

56. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}-\dfrac{1}{(c-a)(c-b)}.$

HINT. Apparently $(a-b)(a-c)(b-c)(b-a)(c-a)(c-b)$ is the L.C.D.; multiplying both terms of the second fraction once by -1 and those of the third twice by -1 the three fractions become

$$\frac{1}{(a-b)(a-c)}+\frac{-1}{(b-c)(a-b)}-\frac{1}{(a-c)(b-c)},$$

of which the L.C.D. is $(a-b)(a-c)(b-c)$.

57. $\dfrac{2}{(x-y)(x-z)}-\dfrac{1}{(x-y)(z-x)}.$

58. $\dfrac{3a}{(a-3)(a-4)}-\dfrac{2a}{(3-a)(4-a)}.$

59. $\dfrac{2}{(m-n)(m+n)}+\dfrac{3}{(n-m)(m-7)}-\dfrac{4}{(n-m)(7-m)}.$

60. $\dfrac{c}{c^2-10c+24}-\dfrac{1}{6c-c^2-8}+\dfrac{3}{(6-c)(2-c)}.$

65. Reduction of a fraction to a mixed expression. A mixed expression is an expression consisting of a rational, integral part and a rational, fractional part.

In arithmetic $4\frac{2}{5}$ means $4+\frac{2}{5}$, while in algebra $a\dfrac{b}{c}$ means a times $\dfrac{b}{c}$, or $\dfrac{ab}{c}$. Hence in algebraic mixed expressions the integral and fractional portions must be connected by a plus or a minus sign.

If the numerator of a fraction (in its lowest terms) is of the same degree as the denominator, or of a higher degree, the fraction may often be reduced to a mixed expression.

Obviously such fractions as $\frac{x^2}{a^2+b^2}$, $\frac{xy}{x^2+y^2}$, and $\frac{x^3+y^3}{xy}$ cannot be reduced to mixed expressions.

Example: Reduce $\frac{4x^4-6x^3-x^2-1}{x^3}$ to a fraction.

Solution: Dividing, $\frac{4x^4-6x^3-x^2-1}{x^3}=4x-6+\frac{-x^2-1}{x^3}$

$$=4x-6-\frac{x^2+1}{x^3}.$$

To reduce a fraction (in its lowest terms) to a mixed expression we have the

Rule. *Perform the indicated division, thus obtaining a partial quotient, until the remainder is of lower degree than the divisor. Write the remainder over the divisor and connect the resulting fraction by a plus sign to the partial quotient, thus forming the complete quotient.*

The reduction of a mixed expression to a fraction is performed as in Exercises 31–41, page 145.

EXERCISES

Reduce to mixed expressions:

1. $\frac{15x^2-10x+2}{5x}$.
2. $\frac{24a^3-6a^2-14}{6a}$.
3. $\frac{c}{c+1}$.
4. $\frac{d^3+1}{d-1}$.
5. $\frac{27x^3-y^3}{3x+y}$.
6. $\frac{y^4+y^2+1}{y^2-y-1}$.
7. $\frac{a^4+a^2b^2+b^4}{a^2+ab-b^2}$.
8. $\frac{3y^3-11}{y+3}$.
9. $\frac{(a+b)^3}{a^3+b^3}$.
10. $\frac{16a^4+b^4}{2a-1}$.
11. $\frac{5a^3+3a^2-6}{5a^2+3a+2}$.
12. $\frac{x^6+y^6}{x^2-y^2}$.
13. $\frac{x^4}{x+1}$.
14. $\frac{(a^2+b)^2}{a^2+b^2}$.

66. Multiplication of fractions. In algebra as in arithmetic the product of two or more fractions is the product of their numerators divided by the product of their denominators.

Thus $\frac{3}{4}\cdot\frac{5}{7}=\frac{15}{28}.$

Similarly $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd},$

and $5\cdot\frac{3}{7}=\frac{5}{1}\cdot\frac{3}{7}=\frac{15}{7}.$

In like manner $n\cdot\frac{a}{b}=\frac{n}{1}\cdot\frac{a}{b}=\frac{na}{b}.$

EXAMPLES

1. Multiply $\frac{4\,a^2x^3}{5\,y^3}$ by $\frac{35\,y^5}{16\,a^3x^2}.$

Solution: Canceling common factors,

$$\frac{4\,a^2x^3}{5\,y^3}\cdot\frac{35\,y^5}{16\,a^3x^2}=\frac{\cancel{4}\,\cancel{a^2}\overset{x}{\cancel{x^3}}}{\cancel{5}\,\cancel{y^3}}\cdot\frac{\overset{7\,y^2}{\cancel{35}\,\cancel{y^5}}}{\underset{4\,a}{\cancel{16}\,\cancel{a^3}\cancel{x^2}}}=\frac{7\,xy^2}{4\,a}.$$

2. Simplify $\frac{x^2-4}{x^2-5\,x+6}\cdot\frac{x+2}{x+3}\cdot\frac{2\,x^3-18\,x}{x^2+4\,x+4}.$

Solution: Factoring and canceling common factors,

$$\frac{x^2-4}{x^2-5\,x+6}\cdot\frac{x+2}{x+3}\cdot\frac{2\,x^3-18\,x}{x^2+4\,x+4}$$

$$=\frac{\cancel{(x+2)}\cancel{(x-2)}}{\cancel{(x-3)}\cancel{(x-2)}}\cdot\frac{\cancel{x+2}}{\cancel{x+3}}\cdot\frac{2\,x\cancel{(x+3)}\cancel{(x-3)}}{\cancel{(x+2)}\cancel{(x+2)}}$$

$$=2\,x.$$

Check: Let $x=1$.

$$\frac{x^2-4}{x^2-5\,x+6}\cdot\frac{x+2}{x+3}\cdot\frac{2\,x^3-18\,x}{x^2+4\,x+4}=2\,x.$$

$$\frac{1-4}{1-5+6}\cdot\frac{1+2}{1+3}\cdot\frac{2-18}{1+4+4}=2.$$

$$\frac{-3}{2}\cdot\frac{3}{4}\cdot\frac{-16}{9}=2.$$

$$2=2.$$

At this point the student should read the rule on page 150 and solve Exercises 1–16, page 151. He should then study Example 3 and solve the remaining exercises.

3. Simplify

$$\left(3a-\frac{b^2-3ab^2}{3a-b^2}\right)\cdot\frac{8a^2y^3}{3ab^2+6a^2b-9a^3}\cdot(a^2-b^2)\cdot\frac{9bx}{16a^3y}.$$

Solution: Reducing the mixed expression to a fraction, this becomes

$$\frac{9a^2-b^2}{3a-b^2}\cdot\frac{8a^2y^3}{3ab^2+6a^2b-9a^3}\cdot(a^2-b^2)\cdot\frac{9bx}{16a^3y}.$$

Factoring and canceling common factors,

$$\frac{\cancel{(3a+b)}(3a-b)}{3a-b^2}\cdot\frac{\cancel{2^3}\cdot\overset{\cancel{a}}{\cancel{a^2}}\cdot\overset{y^2}{\cancel{y^3}}}{\cancel{3}\cancel{a}\underset{-1}{\cancel{(b-a)}}\cancel{(b+3a)}}\cdot\frac{(a+b)\cancel{(a-b)}}{1}\cdot\frac{\overset{3}{\cancel{3^2}}bx}{\underset{2a^2}{\cancel{2^4}\cancel{a^3}\cancel{y}}}.$$

(Note that $(a-b)$ is contained in $(b-a)$, -1 times.)

We have left $\dfrac{(3a-b)y^2(a+b)3bx}{(3a-b^2)(-1)(2a^2)}=\dfrac{9a^2bxy^2+6ab^2xy^2-3b^3xy^2}{-6a^3+2a^2b^2}.$

Check: Let $x=y=a=2$, and $b=3$.

$$\left(3a-\frac{b^2-3ab^2}{3a-b^2}\right)\cdot\frac{8a^2y^3}{3ab^2+6a^2b-9a^3}\cdot(a^2-b^2)\frac{9bx}{16a^3y}$$
$$=\frac{9a^2bxy^2+6ab^2xy^2-3b^3xy^2}{-6a^3+2a^2b^2}.$$

$$\left(6-\frac{9-54}{6-9}\right)\cdot\frac{256}{54+72-72}\cdot(4-9)\cdot\frac{54}{256}$$
$$=\frac{864+864-648}{-48+72}.$$

$$(-9)\cdot\tfrac{256}{54}\cdot(-5)\cdot\tfrac{54}{256}=\tfrac{1080}{24}.$$
$$45=45.$$

To find the product of two or more fractions or mixed expressions we have the

RULE. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

EXERCISES

Simplify:

1. $\frac{3x^2y^3}{4a^4} \cdot \frac{8a^3}{12xy}$.

2. $\frac{10a^3b^2}{15c^3} \cdot \frac{18c}{7a^2}$.

3. $\frac{3x}{8y} \cdot \frac{6ax}{9b} \cdot \frac{12b^2y}{16x}$.

4. $\frac{14m^2n^4}{5n} \cdot \frac{25n^3}{6m^2} \cdot \frac{4x}{7m}$.

5. $\left(\frac{2x}{3ay}\right)^2 \cdot \frac{12a^3}{16b^2xy^3} \cdot 8b^4y^2$.

6. $\frac{a}{4c^2} \cdot \left(\frac{c}{2a}\right)^3 \cdot 16$.

7. $\left(\frac{2a}{c}\right)^2 \cdot \left(\frac{2c}{a}\right)^3 \cdot \left(\frac{1}{4}\right)^2$.

8. $\left(\frac{-3a}{3}\right)^2 \cdot \frac{c^5}{(2a)^5} \cdot \left(\frac{2}{3}\right)^2$.

9. $\frac{6c}{a} \cdot \left(\frac{9c^2}{4a}\right)^2 \cdot \left(\frac{-2a}{3c}\right)^3$.

10. $\frac{(4ax)^2}{225c^4} \cdot \frac{(5ac^2)^3}{(2x^2)^3} \cdot \left(\frac{-x}{c}\right)^5$.

11. $\frac{2a+4x}{5y^2} \cdot \frac{15y}{a+2x}$.

12. $\frac{3c}{9-c^2} \cdot \frac{c^2+5c+6}{18cd}$.

13. $\frac{3a-6y}{4a+2y} \cdot \frac{8(2a+y)^2}{4a^2-24ay+24y^2}$.

14. $\frac{c^2+6ce+9e^2}{6(c+d)^2} \cdot \frac{3c^2e-12d^2e}{c^2-2cd+3ce-6de}$.

15. $\frac{a^2+4ab+4b^2}{9-a^2} \cdot \frac{a^2-5a+6}{a^2-4b^2}$.

16. $\frac{5c^2-20d^2}{c^3+8d^3} \cdot \frac{c^2-2cd+4d^2}{25cd^4}$.

17. $\frac{(x+2)^2}{3x^3+6x^2+12x} \cdot (15-9x^2) \cdot \frac{x^3-8}{4-x^2}$.

18. $\frac{8c-24d}{c^2-6cd+9d^2} \cdot \left(4c-\frac{d^2}{c}\right) \cdot \frac{4c^2+d^2}{64c^4-4d^4}$.

19. $\left(\frac{6m-9}{2m-3}+2m\right)\left(2m-9+\frac{36}{2m+3}\right)$.

20. $\left(\frac{3}{2a}-1\right)^2 \cdot \frac{8a^2x}{9-4a^2} \cdot \frac{3+2a}{2a-3}$.

21. $\left(1+\frac{2}{x}-\frac{3}{x^2}\right) \cdot \frac{9x^3}{3(x^2-18x+17)}$.

22. $\dfrac{x^3-8}{6\,xy}\cdot\left(3-\dfrac{4}{2-x}\right)\cdot\dfrac{18\,x^2y^2}{x^2+2\,x+4}.$

23. $\dfrac{a^3-b^3}{(a-b)^3}\cdot\dfrac{6\,a+6\,b}{2\,a^2+2\,ab+2\,b^2}\cdot\dfrac{(a-b)^2}{a^2-b^2}.$

24. $\left(4-\dfrac{20-9x}{5-x}-x\right)\left(\dfrac{1}{x}+\dfrac{1}{x^2}-\dfrac{30}{x^3}\right).$

25. $\left(x+2\,y-\dfrac{5\,x+10\,y}{x+y}\right)\left(\dfrac{y^2+x^2+2\,yx}{x^2-3\,xy-4\,y^2}\right)\cdot\dfrac{1}{x+2\,y}.$

67. Division of fractions. In arithmetic $\frac{3}{4}\div\frac{5}{7}=\frac{3}{4}\cdot\frac{7}{5}=\frac{21}{20}$; and $\frac{3}{4}\div 11=\frac{3}{4}\cdot\frac{1}{11}=\frac{3}{44}$. Also $\frac{3}{4}\div 1\frac{4}{7}=\frac{3}{4}\div\frac{11}{7}=\frac{3}{4}\cdot\frac{7}{11}=\frac{21}{44}$.

Similarly $\dfrac{a}{b}\div\dfrac{c}{d}=\dfrac{a}{b}\cdot\dfrac{d}{c}=\dfrac{ad}{bc}$; and $\dfrac{a}{b}\div n=\dfrac{a}{b}\cdot\dfrac{1}{n}=\dfrac{a}{bn}.$

Also $\dfrac{a}{b}\div\left(c+\dfrac{n}{d}\right)=\dfrac{a}{b}\div\left(\dfrac{cd+n}{d}\right)=\dfrac{a}{b}\cdot\dfrac{d}{cd+n}=\dfrac{ad}{bcd+bn}.$

For division of fractions we have the

RULE. *Reduce all integral or mixed expressions to fractional form.*

Then invert the divisor, or divisors, and proceed as in multiplication of fractions.

EXERCISES

Perform the indicated operations:

1. $\dfrac{6\,a}{3\,b}\div\dfrac{4\,a}{12\,b^2}.$

2. $\dfrac{9\,a^2}{6\,b^2}\div\dfrac{14\,ab}{8\,x}.$

3. $\left(\dfrac{2\,ax}{3\,c}\right)^2\div\dfrac{6\,ax^2}{9\,c^4}.$

4. $\dfrac{a}{b}\div\dfrac{c}{d}\div\dfrac{ad^2}{bc^2}.$

5. $2\,a\cdot\left(\dfrac{2\,a^2}{3\,x}\right)^2\div\dfrac{(4\,a^2)^2}{18\,x^2}.$

6. $\dfrac{5\,x}{5\,y}\div\dfrac{10\,x^2}{12\,y^3}\div\dfrac{4\,ax}{3\,by^2}.$

7. $\dfrac{6}{(2\,x)^2}\div\dfrac{12}{(4\,x^2)^3}\cdot\left(\dfrac{2}{x}\right)^2.$

8. $\dfrac{3\,m^2}{5\,n}\div\dfrac{21\,m^3}{10\,mn^2}\div 14\,m^4n.$

9. $\left(\dfrac{4\,x^2}{3\,a}\right)^2\div\left(\dfrac{2\,x}{3\,a}\right)^3\cdot\left(\dfrac{a}{x}\right)^2.$

10. $\dfrac{6\,a-3}{5\,x}\div\dfrac{2\,a-1}{15\,bx^3}.$

11. $\dfrac{4-x^2}{3\,x^2+x^3}\div\dfrac{4-4\,x+x^2}{x^3-x^2-12\,x}.$

12. $\frac{a^2-7a+12}{a-1}\div\frac{a^2-16}{1-a^2}$.

13. $(-14-5c+c^2)\div\frac{ac^2-49a}{bc^2+9bc+14b}$.

14. $\frac{4a^2-4ab-3b^2}{8a^3x}\div\left(a-\frac{9b^2}{4a}\right)$.

15. $\left(\frac{2a}{x}\right)^4\div\left(\frac{-6a^2c}{5x}\right)^3\cdot\left(\frac{3c^3}{10ax}\right)^2$.

16. $\left(\frac{-a^2}{x}\right)\div\left(\frac{-a}{x^2}\right)^5\cdot\left(\frac{a}{x}\right)^3\cdot\left(\frac{-1}{x^2}\right)^3$.

17. $\left(\frac{m^2}{n^2}-\frac{n^2}{m^2}\right)\div\left(\frac{m^4+2m^2n^2+n^4}{m^2n-4mn^2}\right)$.

18. $\frac{9x^2+6xy-8y^2}{2x+y}\div(3x-2y)^2\div\left(2-\frac{x-2y}{2x+y}\right)$.

19. $\left(\frac{a}{c}+\frac{c}{a}\right)\div\left(\frac{a^6+c^6}{a^3c^3}\right)\left(a^2-c^2+\frac{c^4}{a^2}\right)\cdot\frac{1}{ac^2}$.

20. $\frac{2a-5}{2a^2+2}\div\frac{4a^4-25a^2}{4-4a^4}\cdot\left(3a^2-\frac{a^3-8a^2}{a-1}\right)$.

21. $\frac{8m^3-125n^3}{m^2+mn}\div\left(\frac{2m}{5n}-\frac{5n}{2m}\right)\div\left[3mn\left(2m+5n+\frac{25n^2}{2m}\right)\right]$.

22. $\left(\frac{4y}{x}-\frac{15y^2}{x^2}+4\right)\div\left(4-\frac{16y}{x}+\frac{15y^2}{x^2}\right)\left(3-\frac{4x+20y}{2x+5y}\right)$.

23. $\left(\frac{9c^2-4d^2}{6c^2}\right)\div\left(d+4c+\frac{15c^2}{4d}\right)\cdot\frac{15c^2d+6cd^2}{9c^2d+24cd^2-20d^3}$.

24. $\left(6x-11-\frac{7}{x}\right)\div\left(2+\frac{11}{x}+\frac{5}{x^2}\right)\div\left(\frac{1}{3x^3-75x}\right)$.

25. $\left(x^2-y^2+\frac{4xy(y+x)}{x-y}\right)\div\left(\frac{x^2+y(y+2x)}{2x^2-3xy+y^2}\right)$.

68. Complex fractions. A **complex fraction** is a fraction containing a fractional expression either in its numerator or in its denominator or in both.

EXAMPLE

Simplify $\dfrac{x-3-\dfrac{10}{x}}{1-\dfrac{2}{x}-\dfrac{15}{x^2}}$.

Solution: Reducing the numerator and the denominator to simple fractions,

$$\frac{x-3-\dfrac{10}{x}}{1-\dfrac{2}{x}-\dfrac{15}{x^2}}=\frac{\dfrac{x^2-3x-10}{x}}{\dfrac{x^2-2x-15}{x^2}}.$$

Performing the indicated division,

$$\frac{\dfrac{x^2-3x-10}{x}}{\dfrac{x^2-2x-15}{x^2}}=\frac{\cancel{(x-5)}(x+2)}{\cancel{x}}\cdot\frac{\overset{x}{\cancel{x^2}}}{\cancel{(x-5)}(x+3)}$$

$$=\frac{x^2+2x}{x+3}.$$

Check: Let $x=1$.

Then $\dfrac{1-3-\frac{10}{1}}{1-\frac{2}{1}-\frac{15}{1}}=\dfrac{1+2}{1+3}$, or $\dfrac{-12}{-16}=\dfrac{3}{4}$, or $\dfrac{3}{4}=\dfrac{3}{4}$.

To simplify a complex fraction we have the

Rule. *Reduce both the numerator and the denominator to simple fractions, then perform the indicated division.*

EXERCISES

Simplify:

1. $\dfrac{4-\frac{1}{4}}{2+\frac{1}{2}}$.
2. $\dfrac{\frac{4}{9}+1}{\frac{7}{2}-2}$.
3. $\dfrac{9-(\frac{1}{5})^2}{4-\frac{4}{5}}$.
4. $\dfrac{\frac{3}{5}+\frac{4}{7}}{2-\frac{3}{5}\cdot\frac{4}{7}}$.
5. $\dfrac{(\frac{8}{5})^2-2}{3\cdot\frac{8}{5}}$.
6. $\dfrac{4-\frac{3}{2}+\frac{2}{3}}{3-\frac{2}{3}+\frac{3}{2}}$.
7. $\dfrac{\frac{2^3}{27}+1}{\frac{9}{2}-1}$.
8. $\dfrac{2\frac{1}{2}-3\frac{1}{3}+4\frac{3}{4}}{2\frac{1}{2}\cdot3\frac{1}{3}-4\frac{3}{4}}$.
9. $\dfrac{2-\frac{9}{2}}{(\frac{1}{2})^2-(\frac{1}{2})^3-12(\frac{1}{2})^4}$.
10. $\dfrac{a/b}{c/d}$.

11. $$\frac{\dfrac{1+x}{x}}{1-\dfrac{1}{x^2}}.$$

12. $$\frac{c+\dfrac{c}{d}}{1+\dfrac{1}{d}}.$$

13. $$\frac{9-\dfrac{b^2}{4a^2}}{1-\dfrac{b}{6a}}.$$

14. $$\frac{\dfrac{a^2}{b^2}-\dfrac{b^2}{a^2}}{a+\dfrac{b^2}{a}}.$$

15. $$\frac{a-\dfrac{9}{a}}{\dfrac{1}{a^2}-\dfrac{1}{a^3}-\dfrac{12}{a^4}}.$$

16. $$\frac{\dfrac{m^3}{n^3}+1}{m+\dfrac{n^2}{m}-n}.$$

17. $$\frac{2+\dfrac{1}{a-2}+a}{a+\dfrac{1}{a+2}-2}.$$

18. $$\frac{\dfrac{a}{b}}{c}-\frac{a}{\dfrac{b}{c}}.$$

19. $$\frac{1+\dfrac{b}{a}-\dfrac{20a}{b}}{\dfrac{b}{a}-\dfrac{8a}{b}-2}.$$

20. $$\frac{\dfrac{16}{x}-x}{\dfrac{24}{x^4}+\dfrac{10}{x^3}+\dfrac{1}{x^2}}.$$

21. $$\frac{x+\dfrac{x^2+y^2}{y}}{\dfrac{x}{y}-1}.$$

22. $$\frac{2+\dfrac{a}{b}}{\dfrac{(a-2b)^2}{4ab}+2}.$$

23. $$\frac{\left(\dfrac{2x^2}{x-y}\right)^2}{\dfrac{8x^3}{(x-y)^3}}.$$

24. $$\frac{\dfrac{a-b}{a}-\dfrac{a+b}{b}}{\dfrac{a-b}{b}+\dfrac{a+b}{a}}.$$

25. $$1-\frac{\left(1-\dfrac{c^4}{9}\right)-\left(1-\dfrac{c^4}{16}\right)}{1-\dfrac{7c^4}{144}}.$$

26. $$\frac{\dfrac{a}{1+a}+\dfrac{1-a}{a}}{\dfrac{a}{1+a}-\dfrac{1-a}{a}}.$$

27. $$\frac{\dfrac{9}{8}x-\dfrac{9y}{8}-\dfrac{7y^2}{2x}}{\dfrac{3x}{4}+\dfrac{9}{4}y+\dfrac{5y^2}{3x}}.$$

28. $$\frac{\dfrac{x}{2ax+3bx+6ab+x^2}}{\dfrac{1}{x+2a}-\dfrac{1}{x+3b}}.$$

29. $$\frac{\left(\dfrac{3a+4b}{3a}\right)^2-\dfrac{6b}{a}}{8a-\dfrac{(3a+2b)^2}{3b}}.$$

30. $$\frac{1-\dfrac{x^2-1}{5x^2-6x+1}}{1+\dfrac{3x+2}{5x-1}}.$$

CHAPTER XVII

EQUATIONS CONTAINING FRACTIONS

69. Monomial denominators. Equations containing fractions with monomial denominators are easily solved. Yet unless each fraction preceded by a minus sign is handled with care, errors will be frequent.

EXAMPLE

Solve the equation $\frac{2}{7}\left(9+\frac{5x}{3}\right)-\frac{3(2x-1)}{5}+\frac{5x}{3}=2x.$

Solution: Performing the indicated multiplication,

$$\frac{18}{7}+\frac{10x}{21}-\frac{6x-3}{5}+\frac{5x}{3}=2x.$$

Multiplying each member by the L.C.M. of the denominators, 105, and canceling,

$$\frac{18}{\cancel{7}}\cdot\overset{15}{\cancel{105}}+\frac{10x}{\cancel{21}}\cdot\overset{5}{\cancel{105}}-\frac{6x-3}{\cancel{5}}\cdot\overset{21}{\cancel{105}}+\frac{5x}{\cancel{3}}\cdot\overset{35}{\cancel{105}}=2x\cdot 105.$$

$$270+50x-(6x-3)21+175x=210x.$$

$$270+50x-126x+63+175x=210x.$$

Combining like terms, $99x+333=210x.$

Then $-111x=-333.$

Whence $x=3.$

Check: $\frac{2}{7}\left(9+\frac{15}{3}\right)-\frac{3(6-1)}{5}+\frac{15}{3}=6.$

$$4 - 3 + 5 = 6, \text{ or } 6 = 6.$$

For solving equations containing fractions with monomial denominators, we have the

Rule. *Free the equation of any parentheses it may contain.*

Find the L.C.M. of the denominators of the fractions and multiply each fraction and each integral term of the equation by it, using cancellation wherever possible.

Transpose and solve as usual.

EXERCISES

Find the roots of the following and verify results:

1. $\frac{x}{2}+\frac{x}{3}=10.$

2. $\frac{4}{3}x+\frac{2}{5}x=5\frac{1}{5}.$

3. $\frac{x+5}{4}-\frac{2x+4}{9}=1.$

4. $\frac{2x+3}{5}-\frac{1}{3}(x-3)=2.$

5. $\frac{4x+2}{11}-\frac{1}{5}(x+5)=0.$

6. $\frac{3}{4}(x+1)-\frac{5x-7}{6}=\frac{7}{3}.$

7. $\frac{x+6}{10}+\frac{3}{2}(x+4)=-3.$

8. $\frac{5x-12}{6}-\frac{4}{11}(2x-7)=\frac{1}{3}.$

9. $2x-1-\frac{12x-7}{6}-\frac{1}{6}=0.$

10. $\frac{10x-7}{6}+\frac{5}{2}\left(\frac{2}{5}-x\right)=\frac{15x-11}{3}.$

11. $\frac{5x}{6}-\frac{1}{2}-\frac{3}{8}\left(x-\frac{5}{3}\right)+\frac{7}{32}=0.$

12. $\frac{5}{x}+\frac{4}{3}=\frac{9}{x}.$

13. $\frac{9x}{4}-\frac{3}{4}-\frac{3x-7}{3}+4=-\frac{17}{24}.$

14. $\frac{c}{3x}-\frac{c}{5x}=\frac{1}{15}.$

15. $\frac{1}{2x}-\frac{13}{24}=\frac{8}{3x}.$

16. $\frac{2a-3x}{6a}+\frac{5a-2x}{5a}+\frac{41}{30}=0.$

17. $\frac{x}{a}-\frac{1}{3}(a-3x)+\frac{19a}{3}=-6.$

18. $\frac{cx}{3}-\frac{n}{5}(3x-5cn)=cn\left(\frac{2c}{3}-\frac{n}{5}\right).$

19. $2x - b - \frac{b}{a}(3x - 4b) + 2a = \frac{2(2a^2 - b^2)}{a}$.

20. $(x + 5)(x - 6) = x(x - \frac{5}{2})$.

21. $\left(x + \frac{3}{4}\right)\left(3 + \frac{x}{2}\right) = \frac{x}{2}(x - 5) + 8\frac{1}{8}$.

22. $(x - \frac{1}{2})(x + \frac{3}{7}) = (x - 1)(x + 2) + 1\frac{1}{14}$.

23. $(x - \frac{3}{2})(x + \frac{2}{3}) - (x - \frac{1}{3})^2 - 1\frac{1}{9} = 0$.

24. $(x + \frac{2}{5})^2 - (x - \frac{1}{2})(x + \frac{1}{5}) + \frac{31}{50} = 0$.

PROBLEMS

1. One fourth of a certain number plus $\frac{1}{12}$ of that number equals 16. Find the number.

2. The difference between $\frac{1}{3}$ of a certain number and $\frac{1}{17}$ of it is 70. Find the number.

3. The sum of two numbers is 38. One tenth of the greater number equals $\frac{1}{9}$ of the less. Find the numbers.

4. The width of a rectangle is $\frac{4}{5}$ of its length. The perimeter is 216 centimeters. Find the area of the rectangle.

5. What number must be added to the numerator of the fraction $\frac{4}{7}$ so that the resulting fraction will be $\frac{1}{5}$ of the number?

6. Three fourths of a certain number is $\frac{1}{3}$ the sum of the next two consecutive numbers. Find the numbers.

7. A certain odd number divided by 11 is equal to $\frac{1}{24}$ of the sum of the next two consecutive odd numbers. Find the numbers.

8. What number added to both terms of the fraction $\frac{10}{17}$ gives a fraction whose value is $\frac{5}{7}$?

9. Separate 42 into two parts such that $\frac{1}{6}$ of their difference is $\frac{1}{3}$.

10. One fourth the difference of three times a certain number and 4 equals $\frac{1}{7}$ the difference of five times the number and 4. Find the number.

11. Separate 112 into two parts such that their quotient is $\frac{2}{5}$.

12. There are two numbers whose sum is 24. If their difference be divided by their sum, the quotient will be $3\frac{5}{6}$ less than the difference of the two numbers. Find the numbers.

13. The quotient of 27 plus seven times a certain number, divided by twice the number equals the quotient of 90 plus five times the number, divided by three times the number. Find the number.

14. A's age is $\frac{5}{2}$ B's age. In 10 years A's age will be twice B's age. Find their ages now.

15. The age of A is $\frac{2}{3}$ that of B. Fourteen years ago A's age was $\frac{1}{2}$ B's age. Find their ages now.

16. A is 16 years older than B. Eight years ago B was $\frac{3}{5}$ as old as A. Find their ages now.

17. Jupiter has 4 more moons than Uranus, and Saturn 2 more than twice as many as Uranus; Mars has 6 fewer than Jupiter, and Neptune half as many as Mars. These planets have together 25 moons. How many has each?

18. A triangle has the same area as a trapezoid. The altitude of the triangle is 30 meters and its base is 8 meters. The altitude of the trapezoid is $\frac{1}{3}$ that of the triangle, and one base equals the base of the triangle. Find the other base of the trapezoid.

19. A marksman hears the bullet strike the target 3 seconds after the report of his rifle. If the average velocity of the bullet is 1925 feet per second and the velocity of sound is 1100 feet per second, find the distance to the target and the length of time the bullet was in the air.

20. A gunner using one of the best modern rifles would hear the projectile strike the target 2640 yards distant in $9\frac{2}{5}$ seconds after the report of the gun, provided the projectile maintained throughout its flight the same velocity it had on leaving the gun. Find this velocity if sound travels 1100 feet per second.

70. Equations containing fractions with polynomial denominators. The method of solving equations of this type is illustrated in the examples which follow.

EXAMPLES

1. Solve the equation $\frac{x^2}{2x^3-2}=\frac{2x}{3x^2+3x+3}-\frac{1}{6x-6}$.

Solution: Factoring the denominators and rewriting,

$$\frac{x^2}{2(x-1)(x^2+x+1)}=\frac{2x}{3(x^2+x+1)}-\frac{1}{6(x-1)}.$$

Multiplying both members of the equation by the L.C.M. of the denominators, $6(x-1)(x^2+x+1)$, and canceling,

$$\frac{x^2}{\cancel{2(x-1)(x^2+x+1)}}\cdot\overset{3}{\cancel{6}}\cancel{(x-1)(x^2+x+1)}$$

$$=\frac{2x}{\cancel{3(x^2+x+1)}}\cdot\overset{2}{\cancel{6}}(x-1)\cancel{(x^2+x+1)}-\frac{1}{\cancel{6(x-1)}}\cdot\cancel{6(x-1)}(x^2+x+1).$$

Then $3x^2=4x^2-4x-x^2-x-1.$

Transposing, $3x^2-4x^2+4x+x^2+x=-1.$

Combining like terms, $5x=-1$, or $x=-\frac{1}{5}$.

Check:

$$\frac{\frac{1}{25}}{-\frac{2}{125}-2}=\frac{-\frac{2}{5}}{\frac{3}{25}-\frac{3}{5}+3}-\frac{1}{-\frac{6}{5}-6}.$$

$$\frac{-5}{252}=\frac{-10}{63}-\frac{-5}{36}.$$

Multiplying by 252, $-5=-40+35$ or $-5=-5$.

In solving equations containing fractions with polynomial denominators, the student should write the denominators and their L.C.M. in factored form, as in the preceding solution. With this exception, the rule on page 156 applies to all equations containing fractions.

Whenever both members of an equation are multiplied by *an expression containing the unknown*, roots *may* be introduced by the process. In fact, an apparent root may thus be obtained for a statement which no number whatever can satisfy. Such statements are frequently called "impossible equations," although, strictly speaking, they are not equations at all.

2. Solve $\dfrac{3x-2}{x-2} = \dfrac{4x-4}{x-2} + 1.$ (A)

Solution: $$\frac{3x-2}{\cancel{x-2}}\cancel{(x-2)} = \frac{4x-4}{\cancel{x-2}}\cancel{(x-2)} + 1\cdot(x-2).$$

Then $3x-2 = 4x-4+x-2.$

Transposing, $3x-4x-x = -4-2+2.$

Combining like terms, $-2x = -4.$

Whence $x = 2.$

On attempting to check, the fraction $\dfrac{3x-2}{x-2}$ becomes $\dfrac{4}{0}\cdot$ Since division by zero has no meaning, 2 is not a root of (A), nor can any number be found which is.

The preceding example illustrates the need of checking; for *an equation has a root*, and a false statement in the form of an equation has none. Moreover the example emphasizes the point that any result we obtain from the solution of an equation is a root, not because we obtain it by correctly performing certain operations, as clearing of fractions, transposing, etc., but because it satisfies the original equation.

The reason that no root can be obtained for the statement $\dfrac{3x-2}{x-2} = \dfrac{4x-4}{x-2} + 1$ is because an impossible number relation is implied therein. This can be shown by solving the equation as follows:

Transposing, $$\frac{3x-2}{x-2} - \frac{4x-4}{x-2} - 1 = 0.$$

Reducing the left-hand member to a fraction,
$$\frac{3x-2-4x+4-x+2}{x-2} = 0,$$

or $$\frac{-2x+4}{x-2} = 0.$$

Factoring the numerator and reducing to lowest terms,
$$\frac{-2\cancel{(x-2)}}{\cancel{x-2}} = 0,$$

or $$-2 = 0.$$

That is, the impossible condition that $-2 = 0$ was implied in stating (A).

The reason that the impossible condition appears by this method of solution before we check, is to be found in the fact that we did not multiply both members of the equation by any expression containing x, as we did in the first solution.

EXERCISES

Solve and check:

1. $\dfrac{32}{x} = 5.$

2. $\dfrac{25}{3x} = 5.$

3. $5x + \dfrac{x-2}{4} = 4x + 7.$

4. $3x - \dfrac{x-1}{4} - \dfrac{8x}{5} = 6.$

5. $\dfrac{4}{x} - \dfrac{12+x}{3x} = \dfrac{4}{3}.$

6. $\dfrac{5}{6} - \dfrac{3x-5}{4x} + \dfrac{x-2}{3x} = 0.$

7. $\dfrac{x+5}{5x} - \dfrac{3(x+1)}{x} = 3\frac{1}{5}.$

8. $5x - 4x\left(3 - \dfrac{2}{x}\right) + \dfrac{1}{4} = 3.$

9. $x - \dfrac{3x}{5}\left(\dfrac{10}{x} - 4\right) + 3\frac{3}{5} = 18.$

10. $\dfrac{8x-7}{4x} + 3\frac{1}{2} = \dfrac{\frac{3}{2} - x}{x}.$

11. $\dfrac{2x-3}{3+2x} = 4.$

12. $\dfrac{x-2}{x-3} = \dfrac{15}{16}.$

13. $\dfrac{5}{7x+5} - \dfrac{1}{8} = 0.$

14. $\dfrac{3x^2 - 7x - 4}{4x^2 - 10x - 8} = \dfrac{3}{4}.$

15. $\dfrac{1}{x-2} = \dfrac{3}{x-3}.$

16. $\dfrac{1}{x-3} = \dfrac{x-6}{x+3} + \dfrac{x}{x+3}.$

17. $\dfrac{x-3}{x+4} = \dfrac{x-9}{x+5}.$

18. $\dfrac{x-2}{x+2} + \dfrac{4}{x+2} + 2 = 0.$

19. $\dfrac{3x}{4} - \dfrac{2}{x-2} = \dfrac{3x-2}{4}.$

20. $\dfrac{4}{x-3} + \dfrac{3x+4}{6} = \dfrac{x}{2}.$

21. $\dfrac{2x+3}{x-5} + 4 = \dfrac{x+8}{x-5}.$

22. $\dfrac{x}{4} - \dfrac{5}{4x-12} = \dfrac{2x + \frac{5}{4}}{3}.$

23. $\dfrac{x-4}{x+5} + \dfrac{7}{5} = \dfrac{3}{x+5}.$

24. $\dfrac{x+4}{15x-5} - \dfrac{x}{5} = \dfrac{2x + \frac{20}{3}}{10}.$

25. $\dfrac{4}{3x+6} + \dfrac{5}{7x+14} + 43 = 0.$

26. $\dfrac{7}{4x-12} + \dfrac{47}{220} = \dfrac{-3}{5x-15}.$

27. $\dfrac{4x}{x+3} - \dfrac{6}{2x+6} = \dfrac{10x+11}{3x+9}.$

28. $\dfrac{3}{x-2} = \dfrac{5}{x^2-25} + \dfrac{3x}{x^2-25}.$

29. $\dfrac{1}{x-3} + \dfrac{2}{x+3} = \dfrac{-3}{x^2-9}.$

30. $\dfrac{x+2}{x-2} = \dfrac{10-x^2}{4-x^2} - \dfrac{10}{x^2-4}.$

31. $\dfrac{x-4}{x-5} + \dfrac{x-15}{x+4} = \dfrac{2x^2-10x-1}{x^2-x-20}.$

32. $\dfrac{x+2}{x+3} + \dfrac{x+3}{x+2} = \dfrac{4x+9}{x^2+5x+6}.$

71. Equations containing decimals. The method of solving an equation containing decimals is illustrated in the following examples.

EXAMPLES

1. Solve the equation $.4x + .7 = 9.7 - .05x$.

Solution: Multiplying by 100, $40x + 70 = 970 - 5x.$
Transposing and collecting, $45x = 900.$
Dividing by 45, $x = 20.$
Check: $.4 \times 20 + .7 = 9.7 - .05 \times 20$, $8.7 = 8.7$.

In equations containing fractions, if decimals occur in any denominator, multiply both terms of such fractions by such a power of ten as will reduce the decimals in the denominators to integers. Then clear the equation of fractions and proceed as in the foregoing example.

2. Solve the equation $\dfrac{4x-3.8}{.5} + \dfrac{1.5x}{.38} + 10x = 9.08.$

Multiplying both terms of the first fraction by 10, and both terms of the second fraction by 100,

$$\frac{40x-38}{5} + \frac{150x}{38} + 10x = 9.08.$$

The equation can now be cleared of fractions and then solved as usual.

EXERCISES

Solve and check:

1. $.3x + 4 = .25.$
2. $.15x - .4x = 235x - 2352.5.$
3. $1.3x + 8.24 = -5.26 - 3.2x.$
4. $3x - 1.245x + .6x = 1.5 + .355x.$
5. $3.5x + .0564 - .1x = 4.9128 - .02x.$
6. $.12(2x + .05) - .15(1.5x - 2) = 0.246.$
7. $\frac{.01x + .003}{6} + \frac{.02x + .0008}{7} = .0017.$
8. $\frac{.3(x + 5)}{8} - \frac{4(.25x - .35)}{7} = \frac{14.325}{56}.$
9. $\frac{0.5(6 - .2x)}{.80} - \frac{.3(.4x - 3)}{.16} = 5.$
10. $\frac{.32x}{.05} + \frac{.045x}{.125} = 13.52.$
11. $\frac{33}{x + 5} + \frac{3.75}{.5(x - 8.5)} = 0.$

Note. The introduction into Europe of the Arabic notation for numbers was one of the important events of the Middle Ages. This notation originated among the Hindus at least as early as 700 A.D. It was adopted by the Arabs, and was introduced by the Moors into Spain during the twelfth and thirteenth centuries. Any one who has tried to multiply two numbers in the Roman notation, like MDCCVII by MCXVIII, will realize the difficulties that surrounded arithmetical operations before the Arabic system was taught. Before the introduction of this system, one of the principal uses for arithmetic was the determination of the day of the month on which Easter came. Roger Bacon in the thirteenth century urged the theologians "to abound in the power of numbering," so that they might carry out these computations. Business accounts were kept on the abacus, a contrivance of wires and sliding balls on which arithmetical operations can be performed with great rapidity.

Though computation in the decimal system was common in Europe from the thirteenth century, the final step in perfecting the notation was not taken until about 1600, when Sir John Napier made use of the decimal point in the modern sense. It was not until the beginning of the eighteenth century that it came into general use.

72. Literal equations. At this point the student should review the solution on page 82.

EXERCISES

Solve and check:

1. $5\,cx - 8\,c^2 = 4\,c^2 - cx.$

2. $2(x+1) - 4\,k = 2.$

3. $3(2\,x - a) = 2(x - 2\,a).$

4. $ax + bx = a^2 + ab.$

5. $cx + b^2 = bx + bc.$

6. $mx + n^2 = m^2 - nx.$

7. $6\,ac + cx + 4\,a^2 = 2\,ax + 3\,c^2 + 2\,ca.$

8. $5\,ax - 5\,a^2 + 6\,b^2 = +\,7\,ab + 3\,bx.$

9. $\dfrac{x}{2\,a} = b.$

10. $\dfrac{3\,ab}{x} = a.$

11. $\dfrac{a}{x} + \dfrac{3\,a}{2\,x} = \dfrac{5}{4}.$

12. $\dfrac{4\,a}{3\,x} + \dfrac{4\,a}{x} = \dfrac{3}{2} + \dfrac{5\,a}{6\,x}.$

13. $\dfrac{x}{a} + \dfrac{x}{b} = a + b.$

14. $\dfrac{c^2}{x} - c = \dfrac{d^2}{x} + d.$

15. $\dfrac{ax}{2\,b} - 4\,b^2 = \dfrac{2\,bx}{a} - a^2.$

16. $\dfrac{x}{c} + \dfrac{c - 2\,x}{3} - 3\,c = -\,4.$

17. $\dfrac{x}{a} + \dfrac{x}{c} + ac = bc + ab + \dfrac{x}{b}.$

18. $\dfrac{3\,b + 4\,x}{5\,b} + \dfrac{3\,b + 2\,x}{4\,b} = \dfrac{1}{20}.$

19. $\dfrac{ax}{2} - \dfrac{3\,b}{5}\left(x - \dfrac{2\,ab}{3}\right) = ab\left(\dfrac{a}{2} - \dfrac{b}{5}\right).$

20. $\dfrac{2\,x - 3\,b}{a} + \dfrac{2}{b}\left(\dfrac{3\,x}{2} - a\right) + 5 + \dfrac{9\,b}{a} = \dfrac{4\,a}{b}.$

21. $\dfrac{x - m^2}{x - n^2} = \dfrac{n}{m}.$

22. $\dfrac{x}{c} - \dfrac{d}{c} = \dfrac{x + 2\,c}{d} - 3.$

23. $\dfrac{b}{b\,(b - x)} + \dfrac{3}{a\,(b - x)} + \dfrac{3\,a + 9}{2\,ab} = 0.$

24. $\dfrac{c}{a\,(x + c)} + \dfrac{a}{c\,(x - a)} = \dfrac{c^2 - ac + 2\,a^2}{2\,ac(x - a)}.$

25. $\frac{1}{ab}+1-\frac{ab}{x}+\frac{1}{abx}=0.$ **26.** $\frac{a^2}{bx}+\frac{b^2}{ax}=\frac{1}{a}+\frac{1}{b}.$

27. $\frac{c^2}{dx}-\frac{d^2}{cx}-\frac{3c-3d}{x}=\frac{c-d}{cd}.$

28. $\frac{a+x}{b+x}-\frac{a-x}{b-x}=\frac{2}{x^2-b^2}.$

29. $\frac{\frac{x}{a-b}-a}{\frac{x}{a-b}+a}+1=\frac{2}{a^2+ab+1}.$

30. $\frac{\frac{a}{2}+\frac{x}{3}}{\frac{a}{3}-\frac{x}{2}}+\frac{7a}{4a-6x}=-7.$

31. $\frac{a^2+ac}{x+3c}+\frac{2cx(a+c)}{x^2+5cx+6c^2}=\frac{a^2+2ac+c^2}{x+2c}.$

73. Meaning of primes and subscripts. Different but related values are often represented by the same letter, with smaller figures or letters written at the right and above or below the letter used; as, y', y'', x_0, $4x_3$, t_m^2 t_w. These are read *y prime, y second, x sub zero, 4 x sub three, the square of t sub m, and t sub w* respectively. Primes and subscripts must not be treated as exponents, and the student should carefully note that x_0 and x_3 are as different numerically as a and b.

The notation just explained is very convenient in physics, where L_1 and L_2 may denote different but related lengths; W_1 and W_2 may represent two different weights; and t_0, t_1, and t_2 may mean three unequal but related intervals of time.

Primes are cumbersome and easily confused with exponents; hence subscripts are preferable.

The following equations are taken from algebra, geometry, and physics, where it is often necessary to express one of the quantities (weight, time, distance, etc.) in terms of the others.

EXERCISES

1. Solve for R, $K = 2\pi RH$.

2. Solve for a, $A = \frac{ab}{2}$.

3. Solve for R, $C = 2\pi R$.

4. Solve for r and t, $d = rt$.

5. Solve for a and A, $\frac{a}{A} = \frac{D}{360}$.

6. Solve for C, $\frac{D}{360} = \frac{l}{C}$.

7. Solve for r, $C = \frac{E}{R + r}$.

8. Solve for r and n, $C = \frac{E}{R + nr}$.

9. Solve for r and n, $C = \frac{n \cdot e}{R + nr}$.

10. Solve for F, $C = \frac{5}{9}(F - 32)$.

11. Solve for W_2, $\frac{W_1}{W_2} = \frac{L_1}{L_2}$.

12. Solve for r and t, $A = P(1 + rt)$.

13. Solve for P_2, $\frac{V_1}{V_2} = \frac{P_2}{P_1}$.

14. Solve for n and l, $s = \frac{n(a + l)}{2}$.

15. Solve for a, l, and r, $s = \frac{rl - a}{r - 1}$.

16. Solve for θ, $\frac{D}{180} = \frac{\theta}{\pi}$.

17. Solve for t_1, $V_1 = V_0(1 + .00365\,t_1)$.

18. Solve for b_2, $A = \frac{(b_1 + b_2)a}{2}$.

19. Solve for x, $\frac{a}{b} = \frac{x}{c - x}$.

20. Solve for F, D_1, and D_0, $\frac{1}{F} = \frac{1}{D_1} + \frac{1}{D_0}$.

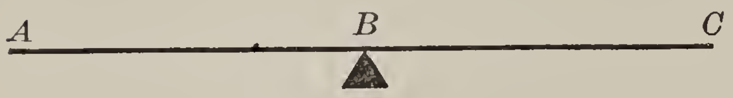
21. $C(t_m - t_1)H_c + W_c(t_m - t_1) = W_h(t_2 - t_m)$.

Find the value of t_m in the preceding equation, when $C = 80$, $t_1 = 20$, $t_2 = 99$, $H_c = .09$, $W_c = 1000$, and $W_h = 800$.

22. $C(t_m - t_w)H_c + W(t_m - t_w) = I(t - t_m)H_I$.

Solve the preceding equation for t, when $C = 80$, $t_m = 54$, $t_w = 18$, $H_c = .09$, $W = 100$, $I = 440$, and $H_I = .11$.

74. The lever. The adjacent figure is a diagram of a machine called a *lever*. AC is a stiff bar resting on a single support at B. This support is called the *fulcrum* and AB and BC are spoken of as *arms* of the lever.



Those who have played with a teeter board have had some experience with a lever, and they have found that, in order to balance, the heavier of two persons must sit nearer the fulcrum than the lighter one does.

Thus, if $AB = 3$ feet and $BC = 4$ feet, a boy at A who weighs 100 pounds will balance a boy at C who weighs 75 pounds; for $3 \cdot 100 = 4 \cdot 75$.

In general, if the length of the arms of a lever are l_1 and l_2 and the corresponding weights are W_1 and W_2, a balance results when

$$l_1W_1 = l_2W_2.$$

PROBLEMS

1. A, 4 feet from the fulcrum, balances B, who is 6 feet from it. A weighs 96 pounds. Find the weight of B.

2. A, who weighs 100 pounds, balances B, who weighs 120 pounds. B is 80 inches from the fulcrum. How far from it is A?

3. A, who weighs 125 pounds, balances B, who weighs 100 pounds. The distance between them is 9 feet. How far is each from the fulcrum?

4. A and B together weigh 210 pounds. They balance when A is 3 feet 9 inches from the fulcrum, and B is 5 feet from it. Find the weight of each.

5. A weighs 90 pounds and is 4 feet from the fulcrum. B weighs 60 pounds and is 3 feet from the fulcrum and on the same side of it as A. C, who weighs 108 pounds, is on the opposite side of the fulcrum. How far from it must C be in order to balance both A and B?

PROBLEMS

1. Separate 300 into two parts such that their quotient is 5.

2. Separate 60 into two parts such that $\frac{2}{3}$ of the greater will equal $\frac{3}{4}$ of the smaller.

3. Separate 45 into two parts such that the sum of $\frac{4}{9}$ of the greater and $\frac{2}{3}$ of the smaller will be 24.

4. Separate $\frac{3}{5}$ into two parts such that $\frac{1}{3}$ of one part will equal $\frac{3}{4}$ of the other.

5. Find two numbers whose sum is 95, such that the greater divided by the less gives a partial quotient of 4 and a remainder of 5.

Solution: Let $x =$ the smaller number.
Then $95 - x =$ the greater number.
Now $\dfrac{\text{Dividend}}{\text{Divisor}} = \text{Partial Quotient} + \dfrac{\text{Remainder}}{\text{Divisor}}.$
Therefore $\dfrac{95 - x}{x} = 4 + \dfrac{5}{x}.$
Multiplying by x, $95 - x = 4x + 5.$
Solving, $x = 18,$
and $95 - x = 77.$

Check:

$$\begin{array}{r|r|l} \underline{18} & 77 & \underline{4} \\ & \underline{72} & \\ & 5 & \end{array}$$

6. Separate 126 into two parts such that one divided by the other gives a partial quotient of 6 and a remainder of 7.

7. The sum of two numbers is 1906. The greater divided by the less gives a partial quotient of 41 and a remainder of 16. Find the numbers.

8. Separate $\frac{8}{5}$ into two parts such that their product is less by $\frac{2}{5}$ than the square of the greater part.

9. Separate 71 into two parts such that 40 exceeds $\frac{2}{3}$ of one part as much as the other exceeds 16.

10. A boy's age is now $\frac{2}{5}$ of what it will be 12 years hence. How old is he?

11. Two thirds of a man's age now, equals $\frac{6}{5}$ of what it was 30 years ago. Find his present age.

12. One sixth of a man's age 8 years ago equals $\frac{1}{8}$ of his age 12 years hence. What is his age now?

13. A man invests part of \$3100 at 6% and the remainder at 5%. The 6% investment yields annually \$18.60 less than the 5% investment. Find the sum invested at 5%.

14. A man invests part of \$5360 at 5% and the remainder at 6%. The yearly income from the 5% investment is \$63.40 more than that from the 6% investment. Find the sum invested at 6%.

15. A part of \$3880 is invested at 4% and the remainder at 6%. The total yearly income is \$171.20. Find the amount invested at 6%.

16. A collection of five-cent pieces and quarters contains 80 coins. Their total value is \$16. How many are there of each?

17. Twenty-eight coins, dimes and quarters, have the value of \$5.05. How many are there of each?

18. The square of half a certain even number is 11 less than $\frac{1}{4}$ the product of the next two consecutive even numbers. Find the numbers.

19. The square of $\frac{4}{3}$ of a certain even number is 2864 less than $1\frac{7}{9}$ times the product of the next two consecutive odd numbers. Find the number.

20. A rectangle is four times as long as it is wide. If it were 4 meters shorter and $1\frac{1}{2}$ meters wider, its area would be 11 square meters more. Find its length and its breadth.

21. The length of a certain rectangle is $2\frac{1}{2}$ times its width. If it were 5 meters longer and 4 meters narrower, its area would be 50 square meters less. Find its dimensions.

22. It costs as much to sod a square piece of ground at 20 cents per square meter as to fence it at 80 cents per meter. Find the side of the square.

23. A rectangular court is twice as long as it is wide. It costs as much to fence it at 50 cents per yard as to sod it at 15 cents per square yard. Find its dimensions.

24. A rectangular picture $2\frac{1}{2}$ times as long as wide is surrounded by a frame 2 inches wide. The area of the frame is 128 square inches. Find the dimensions of the picture.

25. A square court has the same area as a rectangular court whose length is $2\frac{7}{9}$ yards greater and whose width is $2\frac{1}{2}$ yards less. Find the dimensions and area of each.

26. A man bought apples at 18 cents per dozen. He sold $\frac{1}{5}$ of them at the rate of 3 for 4 cents, and the remainder at the rate of 4 for 3 cents, losing 76 cents. How many did he buy?

27. A can do a piece of work in 2 days, B in 3 days, and C in 4 days. How long will it take them, working together?

Solution: Let x represent the number of days required by A, B, and C together to do the work.

Then $\frac{1}{x}$ = the fractional part of the work the three together do in one day.

By the conditions of the problem A does $\frac{1}{2}$ of the work in one day, B does $\frac{1}{3}$ of the work in one day, and C does $\frac{1}{4}$ of the work in one day.

Therefore $$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{1}{x}.$$

Solving, $$x=\tfrac{12}{13}.$$

Check: $$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{1}{\frac{12}{13}}, \text{ or } \frac{13}{12}=\frac{13}{12}.$$

28. A can do a piece of work in $2\frac{1}{2}$ days and B in $3\frac{3}{4}$ days. How many days will they require, working together?

29. A can do a piece of work in 2 days, B in $2\frac{4}{5}$ days, and C in $3\frac{1}{2}$ days. How many days will they require, working together?

30. A can do a piece of work in 8 days, and A and B together can do it in $4\frac{4}{5}$ days. How long would it take B alone?

31. A can do a piece of work in $3\frac{1}{2}$ days, B in $2\frac{4}{5}$ days, and A, B, and C together can do it in $1\frac{1}{6}$ days. In how many days can C do the work alone?

32. A can do a piece of work in 12 days and B in 15 days. After A works 3 days alone, A and B finish the work. How long do they work together?

33. Two bicyclists start at the same time to ride from A to B, 80 miles distant. One travels 4 miles an hour more than the other. The faster bicyclist reaches B and at once returns, meeting the slower one 64 miles from A. Find the rate of each.

Solution: A careful reading shows that the two travel at different rates, that they travel different distances, but that the time is the same for each. Hence the equation must be formed by expressing the time t, or d/r, for each, and equating the two expressions for t.

The two together cover twice the distance from A to B, or 160 miles. As the slower one traveled 64 miles, the faster travels $160-64$, or 96, miles. If x equals the rate of the slower bicyclist in miles per hour, we have:

	d miles	r miles per hour	$\frac{d}{r}$, or t hours
Slower bicyclist	64	x	$\frac{64}{x}$
Faster bicyclist	96	$x+4$	$\frac{96}{x+4}$

Hence
$$\frac{64}{x}=\frac{96}{x+4}.$$

Solving, we obtain $x=8$, the rate of the slower bicyclist in miles per hour, and $x+4=12$, the rate of the faster bicyclist.

Check: $\frac{64}{8}=8$, and $\frac{96}{12}=8$.

34. A man travels at a uniform rate from A to B, 120 miles distant. He travels the first 70 miles without stopping. The remainder of the journey, including a delay of 2 hours, requires the same time as the first part. Find his rate.

Solution: By reading the problem we discover that the distances covered in the first and second portions of the journey are different, that the time of travel is not the same for each, but that the rate throughout is the same. Hence the equation will be formed by finding two expressions for the rate r, or d/t, and setting them equal to each other. If x equals the number of hours required to travel 70 miles, we have:

	d miles	t hours	$\frac{d}{t}$, or r
First part of journey	70	x	$\frac{70}{x}$
Second part of journey	50	$x-2$	$\frac{50}{x-2}$

Hence $$\frac{70}{x}=\frac{50}{x-2}.$$

Solving, we obtain $x=7$, the time in hours occupied in traveling the first 70 miles. And $70 \div 7 = 10$, the rate per hour.

Check: $70 \div 10 = 7$, and $7 - 5 = 2$.

35. A bicyclist traveling 10 miles an hour was overtaken $5\frac{2}{3}$ hours after he started by an automobile which left the same starting point 1 hour and 40 minutes later. What was the rate of the automobile?

36. Two bicyclists, A and B, start at the same time to ride from X to Y, 63 miles distant. A travels 3 miles per hour less than B. The latter reaches Y and at once returns, meeting A 9 miles from Y. Find the rate of each.

37. A leaves a certain point and travels at the rate of $4\frac{1}{2}$ miles an hour. Two and one half hours later B leaves the same point and travels in the opposite direction at the rate of $10\frac{1}{2}$ miles an hour. How much time must elapse after A starts before they will be 40 miles apart?

38. A bicyclist traveled to a certain point and returned, making the trip in 18 hours. The total distance was 160 miles, and the rate going was 8 miles an hour. At what rate did he return?

39. A and B start at the same time from two towns 150 miles apart and travel toward each other. Their respective rates are 9 and 12 miles an hour. A rests 3 hours and B rests $4\frac{1}{2}$ hours before they meet. How far has each of them traveled when they meet?

40. A and B leave the same place at the same time for a point 63 miles distant. A travels $3\frac{1}{2}$ times as fast as B. The former reaches the point and returns immediately, meeting B 8 hours from the time of starting. Find the rates of A and B.

41. A man rows $4\frac{1}{4}$ miles per hour in still water. He finds that it requires 5 hours to row upstream a distance which it requires 3 hours to row down. Find the rate of the current.

HINT. Let x equal the rate of the current. Then $4\frac{1}{4} - x$ equals the rate upstream and $4\frac{1}{4} + x$ equals the rate downstream.

42. A man who can row 4 miles per hour in still water rows up a stream the rate of whose current is 2 miles per hour. After rowing back he finds that the entire trip took 12 hours. How far does he row upstream?

43. A man who can row $4\frac{1}{3}$ miles an hour in still water rows downstream and returns. The rate of the current is $2\frac{1}{4}$ miles per hour and the time required for the trip is 13 hours. How many hours does he require to return?

44. A and B together can do a piece of work in $1\frac{1}{5}$ days. A alone can do the work in one day less than B. Find the time each requires alone.

HINT. Let $x =$ the number of days required by A alone.

Then $x + 1 =$ the number of days required by B alone.

Therefore $\frac{1}{x} + \frac{1}{x+1} = \frac{1}{\frac{6}{5}} = \frac{5}{6}$.

Whence $5x^2 - 7x - 6 = 0$, which may be solved by factoring.

45. A and B together can do a piece of work in $2\frac{2}{5}$ days. B alone can do the work in two days less time than A. Find the number of days required by each alone.

46. A man rows upstream and back, a total distance of 20 miles, in 6 hours. His rate upstream is $2\frac{1}{2}$ miles per hour. Find the rate of the current and his rate in still water.

47. A farmer pays $96 for a number of sheep. He sells all but 2 of them for $100, and gains $2 on each sheep sold. Find the number of sheep bought.

48. A piece of cloth is bought for $64. Four yards are cut off and the remainder is sold, at an advance of $2 per yard, for $72. Find the cost per yard.

49. A train runs 100 miles. On the return trip it increases its rate 5 miles per hour and makes the run in one hour less time. Find the rate going and returning.

50. An automobile makes a run of 120 miles. The chauffeur then increases the speed 4 miles per hour and returns over the same route in 5 hours less time. Find the rate going and returning.

51. Two automobiles travel 72 miles over the same route. One travels 2 miles per hour more than the other and makes the run in 30 minutes less time. Find the rate of each.

52. A and B start at the same time from two points 144 miles apart and travel toward each other. A's rate is 4 miles less than B's. The latter, having been delayed 3 hours on the way, has traveled the same distance as A when they meet. Find the rate of each.

53. A and B start from the same place at the same time and travel in opposite directions. B is delayed 2 hours on the way, and at the end of a certain time the two are 172 miles apart. If A has traveled 28 miles farther than B and one mile more per hour, find the rate of each.

CHAPTER XVIII

RATIO AND PROPORTION

75. Ratio. The **ratio** of one number, a, to a second number, b, is the quotient obtained by dividing the first by the second, or $\frac{a}{b}\cdot$

The ratio of a to b is also written $a : b$.

It follows from the above that all ratios of numbers are fractions and all fractions may be regarded as ratios.

Thus $\frac{3}{2}$, $\frac{c}{2x}$, $\frac{a+b}{a-b}$, and $\frac{\sqrt{2}}{\sqrt[3]{5}}$ are ratios.

The dividend, or numerator, in a ratio is called the **antecedent**, and the divisor, or denominator, is called the **consequent**.

We may speak of the ratio of two concrete numbers if they have a common unit of measure. The ratio of 5 feet to 3 feet is $\frac{5}{3}$, the common unit of measure being 1 foot. Obviously no ratio exists between 5 years and 3 feet.

Measurement is the process of finding the numerical relation (ratio) of whatever is measured to a standard unit of measure. Thus, when we say a distance is 100 yards, we mean that it is 100 times the length of the *standard* yard. For the United States the standard yard is the distance between two scratches on a certain gun-metal bar. This bar, along with the standard pound, the standard gallon, etc., is kept at the Bureau of Weights and Measures in Washington, D.C.

If we say a piece of paper contains 54 square inches, we are expressing by the number 54 the ratio of the surface of the paper to the surface of a square whose side is one inch.

Every measurement, then, is the determination of a ratio, either exact or approximate.

Note. Until comparatively recent times there was no unity among the various nations in regard to the standards of measurement. Just as we now have English, French, and American money, and are obliged to change when we go from one country to another, so until recently the different countries had their own standards of measurement. The yard and the foot are now in common use in English-speaking countries, but in France and Germany the meter is the standard.

In earlier times there was even greater confusion. Among the Hebrews the unit of length was the cubit, which, tradition tells us, was the distance from the end of the king's longest finger to the point of his elbow. Our word *foot* is a reminder of the time when the length of the king's foot was the standard. But with the advance of civilization and the increase of trade between different nations more or less uniformity in standards of measurement has been secured.

EXERCISES

Simplify the following ratios by writing them as fractions and reducing the fractions to their lowest terms:

1. $5 : 10$. **4.** $3\frac{1}{16} : 3\frac{1}{2}$. **7.** 150 lb. : 1 ton.

2. $10 : 5$. **5.** $8\frac{2}{3} : 5\frac{7}{9}$. **8.** $(x^2 - y^2) : (x + y)$.

3. $16\,a^2 : 8\,a$. **6.** 3 days : 9 hours. **9.** $(a^3 + b^3) : (a + b)$.

10. $\left(1 - \frac{1}{a^2}\right) : \left(1 + \frac{1}{a}\right)$. **11.** $\left(2 + \frac{1}{x^2}\right) : \left(4 - \frac{1}{x^4}\right)$.

12. $\frac{1}{x - 3} : \frac{1}{x^2 - 5\,x + 6}$.

13. $(x^2 - xy + y^2) : (x^3 + y^3)$.

14. $\left(x^2 - \frac{y^3}{x}\right) : \left(1 - \frac{y}{x}\right)$.

15. $\left(\frac{16}{x} - x\right) : \left(\frac{24}{x^4} + \frac{10}{x^3} + \frac{1}{x^2}\right)$.

16. Separate 40 into two parts which are in the ratio of $2 : 3$.

Hint. Let $2\,x =$ one part, and $3\,x =$ the other. Then $2\,x + 3\,x = 40$, etc.

17. Separate 16 into two parts which are in the ratio of $5 : 3$.

18. Separate 84 into two parts which are in the ratio of $3:11$.

19. Separate 36 into three parts which are to each other as $2:3:4$.

20. Separate 135 into three parts which are to each other as $4:5:6$.

21. What number added to both terms of the ratio $\frac{5}{8}$ gives as the result the ratio $\frac{21}{28}$?

22. What number subtracted from both terms of the ratio $\frac{13}{27}$ gives as the result the ratio $\frac{2}{9}$?

23. If a is a positive number, which is the greater ratio, $\frac{4+2a}{4+3a}$ or $\frac{4+3a}{4+4a}$?

HINT. Reduce the fractions to equivalent fractions having a common denominator, and then compare the numerators of the resulting fractions.

24. If a and b are positive numbers, which is the greater ratio, $\frac{a+4b}{a+5b}$ or $\frac{a+6b}{a+7b}$?

25. If a positive number is added to both terms of a proper fraction, what change is produced in the numerical value of the fraction?

76. Proportion. Four numbers, a, b, c, and d, are in **proportion** if the ratio of the first pair equals the ratio of the second pair.

This proportion is written $\boldsymbol{a:b=c:d}$, or $\boldsymbol{\frac{a}{b}=\frac{c}{d}}$.

The first and fourth terms (a, d) are called the **extremes,** and the second and third terms (b, c) are called the **means.**

Since a proportion is an equation, any operation which may properly be performed on an equation may be performed on a proportion.

Then in the proportion $\frac{a}{b}=\frac{c}{d}$ both members may be multiplied by bd, giving $\boldsymbol{ad=bc}$.

Therefore, *In any proportion the product of the means equals the product of the extremes.*

EXERCISES

Find the value of x in the proportions:

1. $\frac{3}{4}=\frac{6}{x}\cdot$
2. $\frac{4}{7}=\frac{16}{x}\cdot$
3. $\frac{3}{2}=\frac{x}{5}\cdot$
4. $3:x=7:9.$
5. $x:4=3:6.$
6. $\frac{4}{3}=2:\frac{1}{x}\cdot$
7. $\frac{2}{3}=\frac{\frac{1}{x}}{4}\cdot$
8. $\frac{\frac{2}{1}}{x}=\frac{3\frac{1}{3}}{4}\cdot$
9. $\frac{1}{x}:2=3:4\frac{1}{2}.$
10. $\frac{a}{b}=\frac{c}{x}\cdot$
11. $a:b=c:\frac{1}{x}\cdot$
12. $\frac{a}{x^2}=\frac{1}{x}:a^3.$
13. $4:3\frac{1}{3}=3:x-3.$
14. $5:x-3=7:2x+6.$

A **mean proportional** between two numbers, a and b, is the number m, if $\frac{a}{m}=\frac{m}{b}\cdot$ This means that $m^2=ab$, or $m=\pm\sqrt{ab}.$

Since $\frac{2}{\pm 4}=\frac{\pm 4}{8}$, $+4$ is a mean proportional between 2 and 8, as is also -4.

A **third proportional** to two numbers, a and b, is the number t, if $\frac{a}{b}=\frac{b}{t}\cdot$

In $\frac{3}{6}=\frac{6}{12}$, 12 is a third proportional to 3 and 6.

A **fourth proportional** to three numbers, a, b, and c, is the number f, if $\frac{a}{b}=\frac{c}{f}\cdot$

Since $\frac{5}{12}=\frac{10}{24}$, 24 is a fourth proportional to 5, 12, and 10.

EXERCISES

Find the mean proportionals between:

1. 1 and 4.
2. 4 and 9.
3. 16 and 4.
4. 3 and 12.
5. $(a-b)^2$ and 4.
6. $\frac{1}{2}$ and $\frac{1}{8}$.
7. $\frac{1}{3}$ and $\frac{1}{27}$.
8. $\frac{4}{a^3}$ and $\frac{9}{ax^2}\cdot$

9. Find a third proportional to the numbers in Exercises 1–7 which precede.

Find a fourth proportional to:

10. 1, 2, and 3.

11. 4, 5, and 6.

12. 7, 14, and 5.

13. 5, 12, and a.

14. 7, $21x$, and $6x$.

15. a, a^2, and a^3.

16. a^3, a^5, and a^4.

17. $a+b$, $a-b$, and a^2-b^2.

18. $\frac{a-b}{2}$, $\frac{1}{a+b}$, and a^2-b^2.

If $ps = qr$ is divided by qs, we obtain

$$\frac{p\cancel{s}}{q\cancel{s}} = \frac{\cancel{q}r}{\cancel{q}s}, \text{ or } \frac{p}{q} = \frac{r}{s}. \tag{1}$$

Also $ps = qr$ divided by rs gives

$$\frac{p}{r} = \frac{q}{s}. \tag{2}$$

And $qr = ps$ divided by pr gives

$$\frac{q}{p} = \frac{s}{r}. \tag{3}$$

Therefore, *If the product of any two numbers* (ps) *equals the product of two other numbers* (qr), *one pair may be made the means and the other pair the extremes of a proportion.*

If $\frac{a}{b} = \frac{c}{d}$, then from (1) and (2), $\frac{a}{c} = \frac{b}{d}$. Here $\frac{a}{c} = \frac{b}{d}$ is said to be obtained from $\frac{a}{b} = \frac{c}{d}$ by **alternation.**

If $\frac{a}{b} = \frac{c}{d}$, then from (1) and (3), $\frac{b}{a} = \frac{d}{c}$. Here $\frac{b}{a} = \frac{d}{c}$ is said to be obtained from $\frac{a}{b} = \frac{c}{d}$ by **inversion.**

EXERCISES

Write as a proportion in three ways:

1. $3 \cdot 4 = 2 \cdot 6$.

2. $5 \cdot 6 = 3 \cdot 10$.

3. $3 \cdot 6 = 2 \cdot x$.

4. $a \cdot d = b \cdot c$.

5. $(a+b)(a-b) = 2 \cdot 3$.

6. $(a+b)(a-b) = (a+2)(a+3)$.

Write as a proportion:

7. $a^2 - b^2 = 2 \cdot 3.$

8. $a^2 - 2ab + b^2 = 3 \cdot 6.$

9. $a^2 - 5a + 6 = 4 \cdot 2.$

10. $a^2 - 7a + 12 = (a + b)^2.$

11. $a^2 - 6a + 9 = a^2 - 10a + 16.$

12. $x^2 - 4xy + 4y^2 = a^2 + 10ab + 25b^2.$

13. $xy = 4.$

14. $xy = 3.$

15. $ab = 1.$

16. $abc = de.$

17. $mnp = xyz.$

18. $ab = a + b.$

Write as a proportion so that x is the fourth term:

19. $3 \cdot 4 = 5 \cdot x.$

20. $4x = 9 \cdot 7.$

21. $ab = cx.$

22. $px = qr.$

23. $acx = bd.$

24. $1 = ax.$

25. $x = \dfrac{ab}{c}\cdot$

26. $xy = y + 1.$

Write by alternation:

27. $\frac{2}{3} = \frac{4}{6}.$

28. $\dfrac{4}{3} = \dfrac{x}{y}\cdot$

29. $4 : 5 = 6 : x.$

30. $3 = \dfrac{a}{b}\cdot$

Write by inversion:

31. $4 : 8 = 3 : 6.$

32. $\dfrac{3}{2} = \dfrac{a}{b}\cdot$

33. $P : P_1 = W_1 : W.$

34. $\dfrac{V_1}{V_2} = \dfrac{P_2}{P_1}\cdot$

35. $3 : \dfrac{1}{x} = \dfrac{2}{a}\cdot$

36. $\dfrac{1}{x} : \dfrac{1}{y} = \dfrac{1}{z}\cdot$

37. $\dfrac{N_1}{N_2} = \dfrac{\sqrt{T_2}}{\sqrt{T_1}}\cdot$

If four numbers, a, b, c, and d, are in proportion, they are in proportion by **addition**, **subtraction**, and **addition and subtraction**.

Addition. Let

$$\frac{a}{b} = \frac{c}{d}\cdot \qquad (1)$$

Adding 1 to both members,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \qquad (2)$$

or

$$\frac{\mathbf{a + b}}{\mathbf{b}} = \frac{\mathbf{c + d}}{\mathbf{d}}. \qquad (3)$$

Here (3) is said to be obtained from (1) by *addition*.

Subtraction. Let
$$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

Subtracting 1 from both members,
$$\frac{a}{b}-1=\frac{c}{d}-1, \qquad (2)$$

or
$$\frac{a-b}{b}=\frac{c-d}{d}. \qquad (3)$$

Here (3) is said to be obtained from (1) by *subtraction.*

Addition and Subtraction. Let $\frac{a}{b}=\frac{c}{d}$. (1)

Then
$$\frac{a+b}{b}=\frac{c+d}{d} \text{ (addition)}, \qquad (2)$$

and
$$\frac{a-b}{b}=\frac{c-d}{d} \text{ (subtraction)}. \qquad (3)$$

Dividing (2) by (3),
$$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \qquad (4)$$

Equation (4) is said to be obtained from (1) by *addition and subtraction.*

Addition, subtraction, and addition and subtraction are often called composition, division, and composition and division respectively.

EXERCISES

Write by addition:

1. $\frac{2}{3}=\frac{4}{6}$.

2. $4:12=8:24$.

3. $a:x=1:2$.

4. $4:3=f:x$.

5. $\frac{A_1}{A_2}=\frac{S_1^2}{S_2^2}$.

6. Write Exercises 1–4, preceding, by subtraction.

7. Write Exercises 1–4, preceding, by addition and subtraction.

8. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a+b}{a}=\frac{c+d}{c}$.

9. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a-b}{a}=\frac{c-d}{c}$.

10. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{2a+b}{b}=\frac{2c+d}{d}$.

11. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a+3b}{b}=\frac{c+3d}{d}$.

Write Exercises 12 and 13 by addition and subtraction and solve the resulting equations for x.

12. $\frac{3x+4}{3x-4}=\frac{3+2}{3-2}$. 13. $\frac{6x+3}{6x-3}=\frac{2a+b}{2a-b}$.

A series of equal ratios. If $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}, \tag{1}$$

then $$\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}. \tag{2}$$

Proof: Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=r. \tag{3}$$

Then $$a=br, \tag{4}$$
$$c=dr, \tag{5}$$
$$e=fr. \tag{6}$$

Adding (4), (5), and (6), $$a+c+e=br+dr+fr. \tag{7}$$

Factoring in (7), $$a+c+e=(b+d+f)r. \tag{8}$$

Therefore $$\frac{a+c+e}{b+d+f}=r. \tag{9}$$

Hence, by (3), $$\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}. \tag{10}$$

This result may be expressed verbally: *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

EXERCISES

Test the truth of the preceding result in Exercises 1–4:

1. $\frac{1}{2}=\frac{3}{6}=\frac{5}{10}$.

2. $\frac{1}{a}=\frac{2}{2a}=\frac{3b}{3ab}$.

3. $3:4=6:8=12:16$.

4. $\frac{1}{x-y}=\frac{a}{ax-ay}=\frac{b}{bx-by}$.

5. Taken in the same order, the sides of two triangles are 3, 4, 5, and 9, 12, 15 respectively. What is the ratio of the sides of the first triangle to the corresponding sides of the second? Compare this ratio with the ratio of the perimeter of the first triangle to the perimeter of the second.

6. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove $\frac{2a+3c+4e}{2b+3d+4f}=\frac{a}{b}$.

MISCELLANEOUS EXERCISES IN PROPORTION

1. If $a:b=4:x$, and $a:b=4:\frac{1}{x}$, find x.

2. If $\frac{a}{b}=\frac{9}{y}$, and $\frac{a}{b}=\frac{1}{\frac{1}{y}}$, find y.

3. If $p:r=7:6$, and $q:r=3:5$, find the ratio $p:q$.

4. If $p:q=4:a$, and $q:r=a:7$, find the ratio $p:r$.

5. The sides of a triangle are 8, 10, and 12. The side 12 is divided into the ratio of the other two sides. Find the two parts.

6. The perimeter of a triangle is 63. Two sides are 18 and 24 and the other side is divided in the ratio of these two. Find the two parts of the third side.

7. A flagstaff casts a shadow 12 yards long; at the same time a man 5 feet 10 inches tall casts a shadow 35 inches long. How high is the pole?

Fact from Geometry. If one triangle is similar to another, the sides of the first taken in any order are proportional to the sides of the second taken in the same order.

8. The sides of a triangle are 10, 15, and 20 respectively. In a similar triangle the side corresponding to 10 is 12. Find the other sides. Compare the ratio of the two corresponding sides with the ratio of the perimeters.

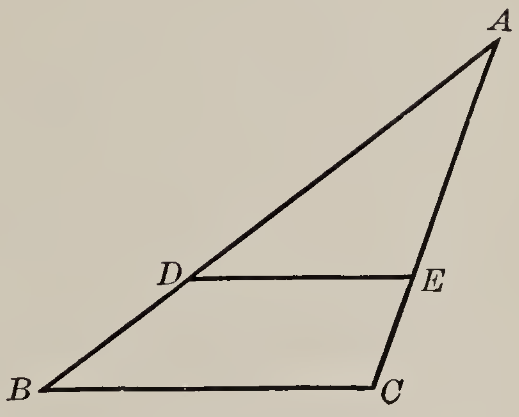
9. The sides of a triangle are 9, 10, and 17. The perimeter of a similar triangle is 108. Find the sides of the second triangle.

Fact from Geometry. A line *parallel* to one side of a triangle divides the other two sides into four proportional parts.

Thus in triangle ABC which follows, line DE is parallel to side BC, and $\frac{AD}{DB}=\frac{AE}{EC}$.

Also a line parallel to one side of a triangle forms with the other two sides a second triangle *similar* to the first.

In the following figure triangle ADE is similar to triangle ABC. Therefore $\frac{AD}{AB}=\frac{AE}{AC}=\frac{DE}{BC}$.



10. In triangle ABC:

(*a*) If $AD = 6$, $DB = 4$, and $AE = 10$, find EC.

(*b*) If $DB = 4$, $AD = 8$, and $DE = 6$, find BC.

(*c*) If $DB = 4$, $AD = 8$, and $AC = 10$, find AE.

(*d*) If $AB = AC = 16$, and $AD = 10$, find AE.

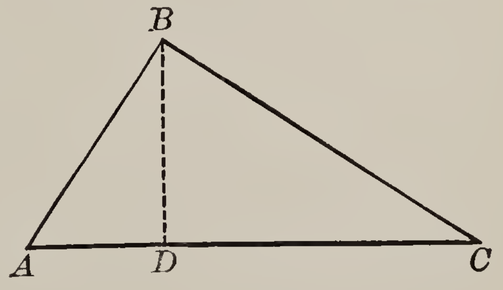
11. Draw a triangle and letter the vertices F, G, and H respectively. Draw RK parallel to FG, R being on side HF and K on side HG. Then if:

(*a*) $FG = 15$, $RK = 10$, $HR = 8$, find HF.
(*b*) $FG = 20$, $RK = 16$, $HF = 10$, find RF.
(*c*) $FG = 20$, $RK = 15$, $RF = 6$, find HR.
(*d*) $FG = 18$, $RK = 15$, $KG = 4$, find HK.

Fact from Geometry. The line joining the middle points of two sides of a triangle is parallel to the third side.

12. Two sides of a triangle are 30 centimeters and 24 centimeters respectively. The line joining their middle points is 12 centimeters long. Find the third side of the triangle.

13. The sides of a triangle are 10, 12, and 16 centimeters respectively. Find the lengths of the lines connecting the middle points of its sides.

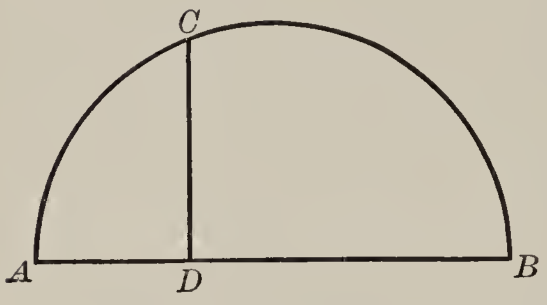


14. In the *right-angled* triangle ABC, line BD is *perpendicular* to AC. Then BD is a mean proportional between AD and DC.

(*a*) If $AD = 9$ and $BD = 6$, find DC.

(*b*) If $AD = 4$ and $AC = 20$, find BD.

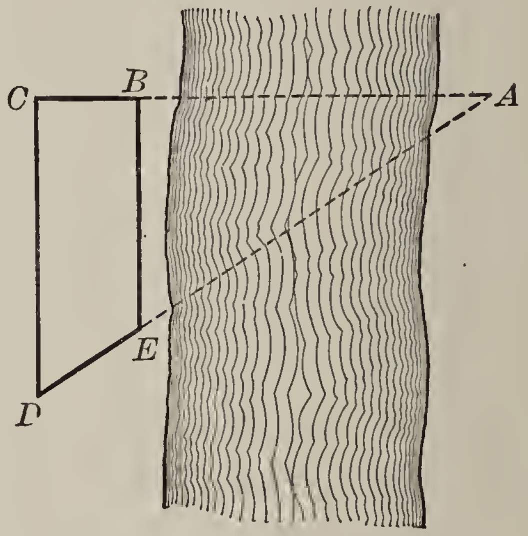
15. In the *semicircle* ABC, line CD is perpendicular to AB. Then CD is a mean proportional between AD and DB.



(*a*). If $AD = 2$ and $DB = 18$, find CD.

(*b*) If $CD = 8$ and $AB = 34$, find AD.

16. The distance AB between two points on opposite banks of a river was wanted. Stakes were set at E, B, D, and C, so that BE was parallel to CD, and so that ABC and AED were straight lines. The measured values of DC, CB, and BE were 480 feet, 160 feet, and 420 feet respectively. What was the computed value of BA?



17. The perimeters of two similar triangles are 45 and 135 respectively. One side of the first is 11 and a second side is 19. Find the sides of the second triangle.

18. Two men start at the same time and travel in opposite directions. The ratio of their rates is 2 : 3. In 5 hours they are 100 miles apart. Find the rate of each.

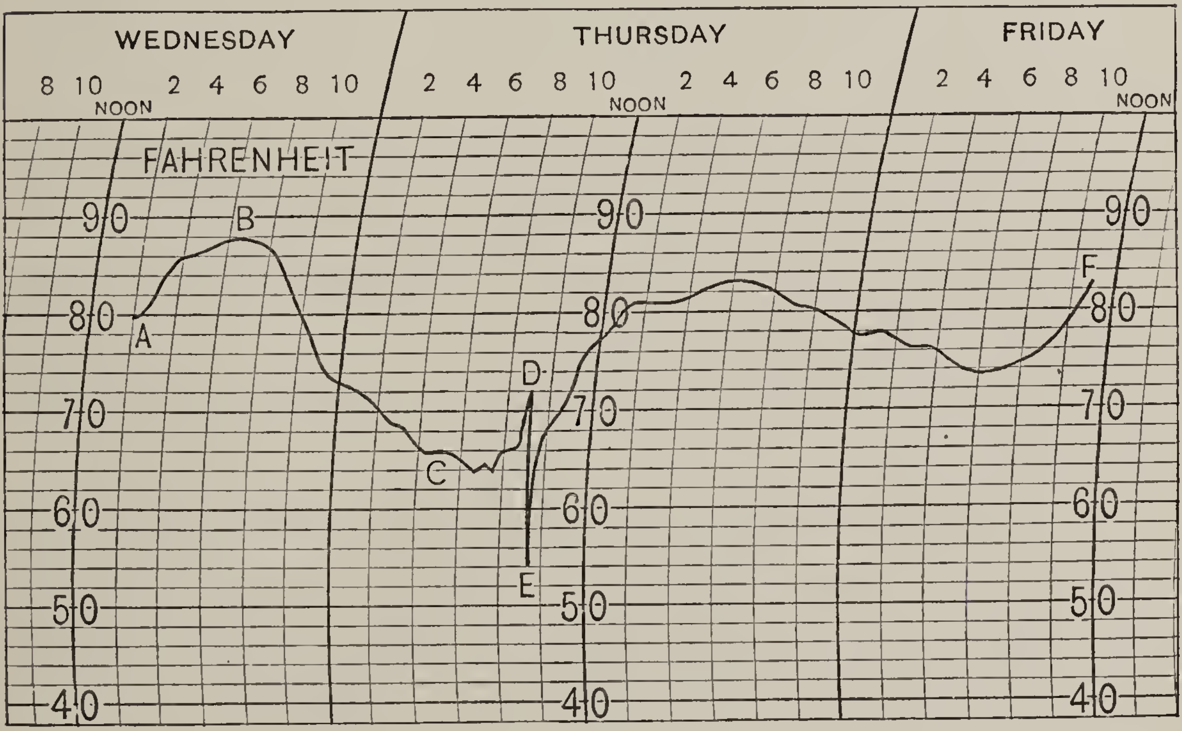
19. A, B, and C ate equally of a stock of provisions which A and B furnished. The values of what A and B contributed were in the ratio of 7 to 8 respectively. C paid $30 for what he ate. How should A and B have divided the money?

20. A clock provided with hands to indicate the minute, the hour, and the day of the month showed correct time at 4 P.M. on February 21, 1900. The clock gained 10 minutes daily. What was the correct time when the clock indicated 4 P.M. on the 28th of the next month?

CHAPTER XIX

GRAPHICAL REPRESENTATION

77. Temperature curve. The curve *ABCDEF* is called a **graph.** It was made by a recording thermometer. Such instruments are provided with an arm carrying a pen, which moves up as the temperature rises, and down as it falls. A clock movement runs a strip of cross-ruled paper under the pen and thus a continuous line is traced on the paper. The following record extends from 2 P.M. of Wednesday, February 19, 1908, to 10.30 A.M. of the Friday following. The numbers 50, 60, 70, 80, and 90 denote degrees Fahrenheit. There are 5 spaces from 50° to 60°. Hence one space corresponds to 2 degrees. The numbers 2, 4, 6, 8, and 10 indicate the time of day. Whether this is A.M. or P.M. can be determined by the position of these numbers with respect to the heavy curved lines marked NOON. The point *A*



on the graph informs us that at 2 P.M. Wednesday the temperature was 80 degrees. The point B between 6 P.M. and 7 P.M. Wednesday marks the highest temperature recorded.

The point C tells us that the temperature was about $65\frac{1}{2}$ degrees at 6 A.M. Thursday.

The preceding record was made indoors, and the sudden fall from D to E was caused by the opening of a door leading into a cold hallway. The portion of the graph from D to E shows that the temperature of the room fell approximately 18 degrees in about 30 minutes.

EXERCISES

By reference to the graph (page 187) answer the following:

1. With what temperature does the record begin? end?

2. What is the highest temperature recorded? the lowest?

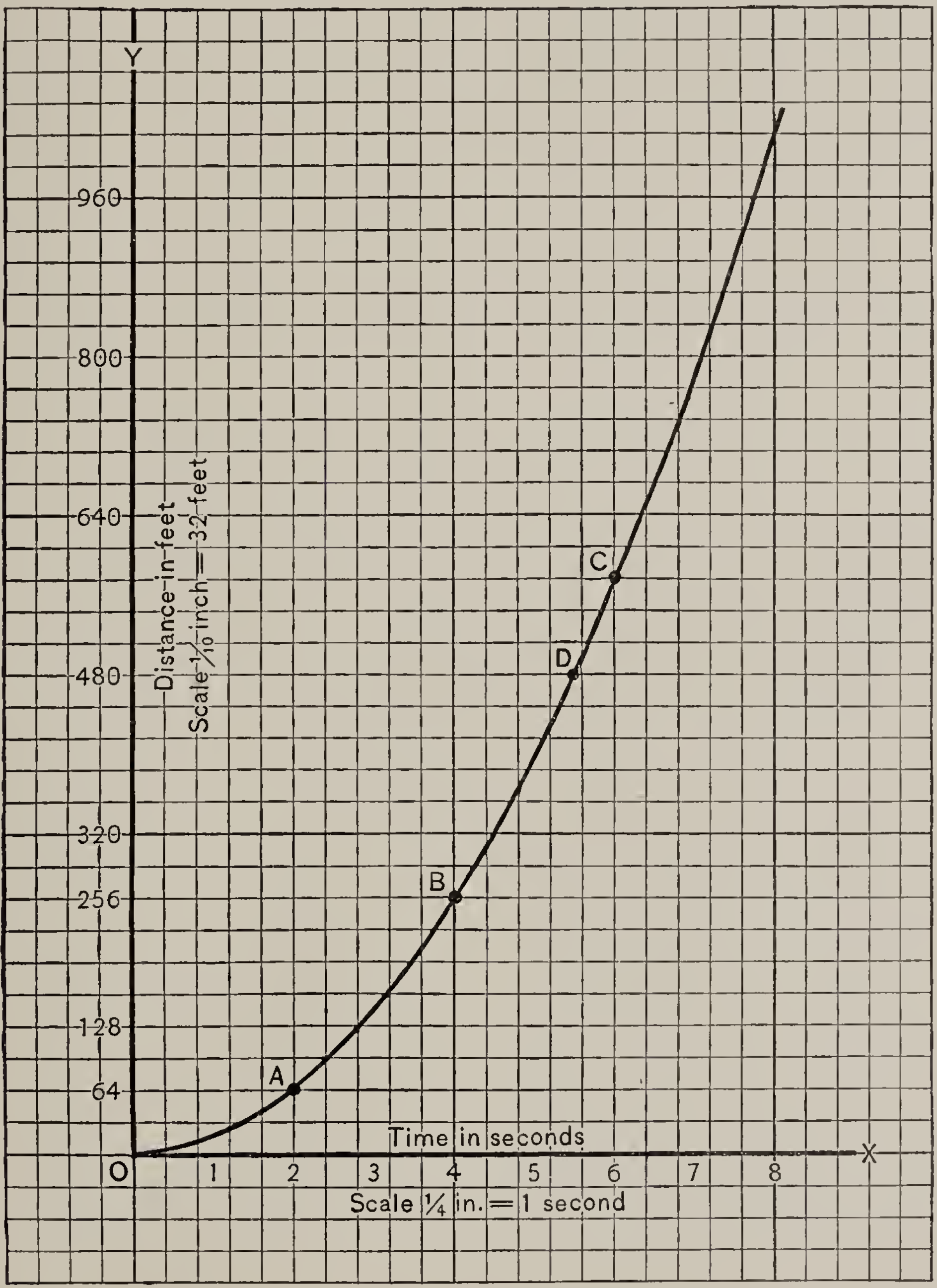
3. About what time was the highest temperature recorded? the lowest?

4. How often did the instrument record a temperature of 80 degrees? 72 degrees? 78 degrees? 62 degrees?

5. At what times did it record a temperature of 80 degrees? 72 degrees? 78 degrees? 62 degrees?

6. To what practical use can a graph such as the one here explained be put?

78. Falling body curve. The curved line $OABDC$ in the adjacent figure is another graph. It represents closely the relation between the distance a sphere of lead, if allowed to drop through the air, will fall in any number of seconds from one to eight. Time measured in seconds from the instant the sphere begins to fall is *represented* on the line OX. One inch on OX corresponds to 4 seconds of time, $\frac{1}{2}$ inch to 2 seconds, $\frac{1}{10}$ inch to $\frac{2}{5}$ of a second, etc. The distance measured in feet through which the sphere falls is *represented* on the line OY. One inch on OY corresponds to 320 feet, $\frac{1}{10}$ of an inch to 32 feet, etc. The point A on the curve, just above 2, corresponds to a time of 2 seconds. *A* is opposite the number 64 on *OY*. This means that in 2 seconds the lead sphere falls 64 feet. Similarly *B* corresponds to a time of 4 seconds and a distance of 256 feet. That is, the lead sphere falls 256 feet in 4 seconds.



The question, "How far does a body fall in 6 seconds?" can be answered by reference to the graph, thus: The point on the curve corresponding to 6 seconds is *C*, just above 6 on *OX*. The point on *OY* opposite *C* corresponds to 576 feet. Therefore in 6 seconds a body falls 576 feet.

EXERCISES

By reference to the graph (page 189) answer the following:

1. How far does a body fall in 8 seconds? 3 seconds? 7 seconds? $2\frac{1}{2}$ seconds? 6.2 seconds?

The question, "How long will it take a body to fall 480 feet?" can be answered by reference to the curve, thus: Opposite 480 on OY is the point D on the curve. D is directly over a point midway between 5 and 6 on OX. Therefore, to fall 480 feet a body requires $5\frac{1}{2}$ seconds.

By reference to the graph, answer the following:

2. How many seconds does a body require to fall 400 feet? 196 feet? 100 feet? 25 feet? 120 feet? 750 feet?

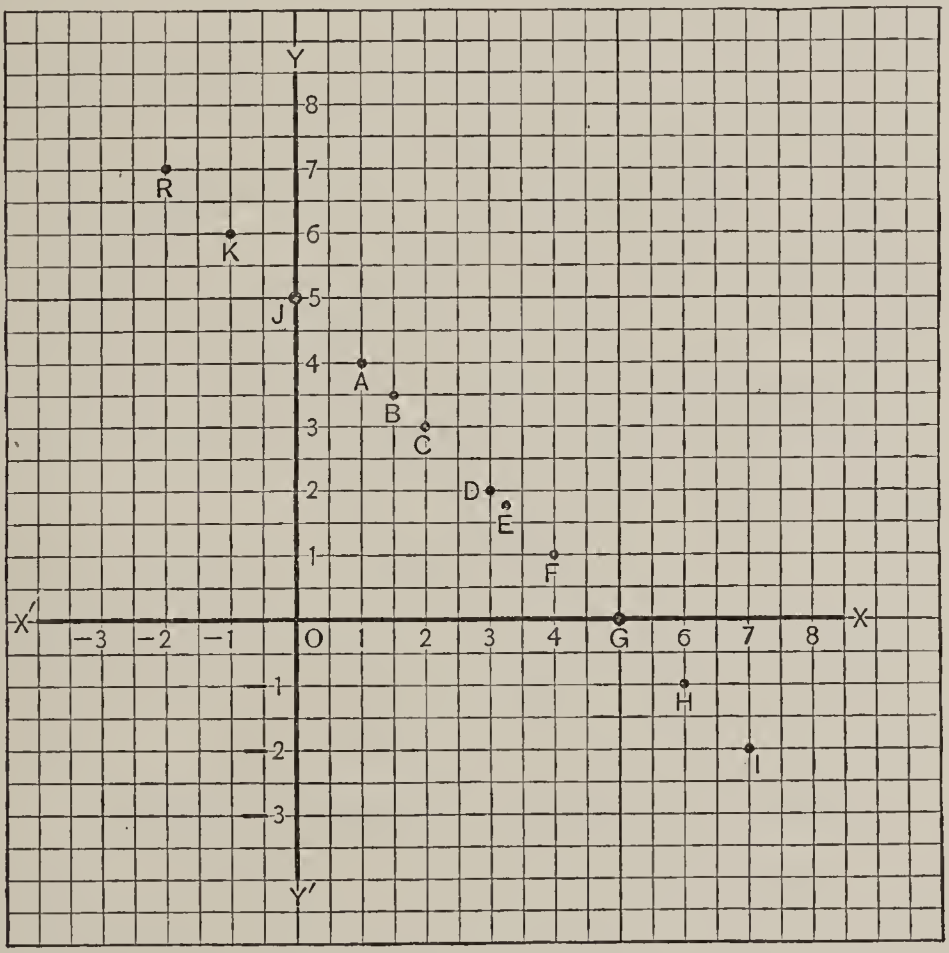
The two preceding graphs are pictorial representations of the relation between two variables. In the first graph the variables were time and temperature, both of which, in the period under consideration, were constantly changing. In the second graph the variables were time and distance. It must be borne in mind that the correctness of any graph is limited by the fact that we cannot measure any physical quantity with perfect accuracy, and that we cannot draw the graph itself with absolute precision. This makes results obtained graphically only approximately correct, but close enough, nevertheless, to be extremely useful for many purposes.

79. Graph of an equation. A relation between two variable numbers not connected with physical quantities, such as temperature and time, can also be represented by a graph. The question, "What two numbers added give five?" may be expressed by the equation $x + y = 5$. Here x and y are *any* two numbers whose sum is 5.

It can be seen by inspection that if x is 1, y is 4, and if x is 2, y is 3. Or we may proceed as follows: Give x any value, say 3; then the equation becomes $3 + y = 5$. Transposing and solving, $y = 2$. Similarly give x the value $3\frac{1}{2}$; then $3\frac{1}{2} + y = 5$, whence $y = 1\frac{1}{2}$. Proceeding in this way, we may obtain a few of the many related pairs of values of x and y, which may be tabulated as follows:

	A	B	C	D	E	F	G	J	H	I	K	R
x	1	$1\frac{1}{2}$	2	3	$3\frac{1}{5}$	4	5	0	6	7	-1	-2
y	4	$3\frac{1}{2}$	3	2	$1\frac{4}{5}$	1	0	5	-1	-2	6	7

Now in the figure we lay off equal spaces on OX from O, and on OY from O each $\frac{2}{10}$ of an inch, and agree to have the values of x correspond to distances measured from OY parallel to OX, and



the values of y to distances measured from OX parallel to OY. Then the point A corresponds to the first pair of numbers, $x = 1$, $y = 4$. In like manner, B corresponds to the second pair of numbers, $x = 1\frac{1}{2}$, $y = 3\frac{1}{2}$. Similarly C, D, E, and F correspond respectively to the third, fourth, fifth, and sixth pairs of numbers.

Apparently A, B, C, etc., are points on a straight line. Even points B and E, corresponding to fractional values of x and y, are in line with the others. The inference seems warranted, then, that a **straight line** drawn from A to F would be a *portion* of the graph of the equation $x + y = 5$.

If the line AF is continued, it meets OX at G. The distance of G to the right of OY is 5 and its distance above OX is zero. Evidently point G corresponds to the seventh pair of numbers, $x = 5$, $y = 0$. Similarly FA extended cuts OY at point J whose distance from OX is 5 and whose distance to the right of OY is zero. Therefore this point corresponds to the eighth pair of numbers, $x = 0$, $y = 5$.

If AF is extended below OX, it passes through points H and I. The point H is just under the sixth space mark on OX and 1 space below OX, and the point I is just under the seventh space mark on OX and 2 spaces below OX. The point H must correspond to the ninth pair of numbers, $x = 6$, $y = -1$, and I to the tenth pair, $x = 7$, $y = -2$. This leads us to extend the line YO **downward** and divide it into spaces equal to those above O, and to number the *consecutive* points of division with the **negative numbers,** -1, -2, -3, etc.

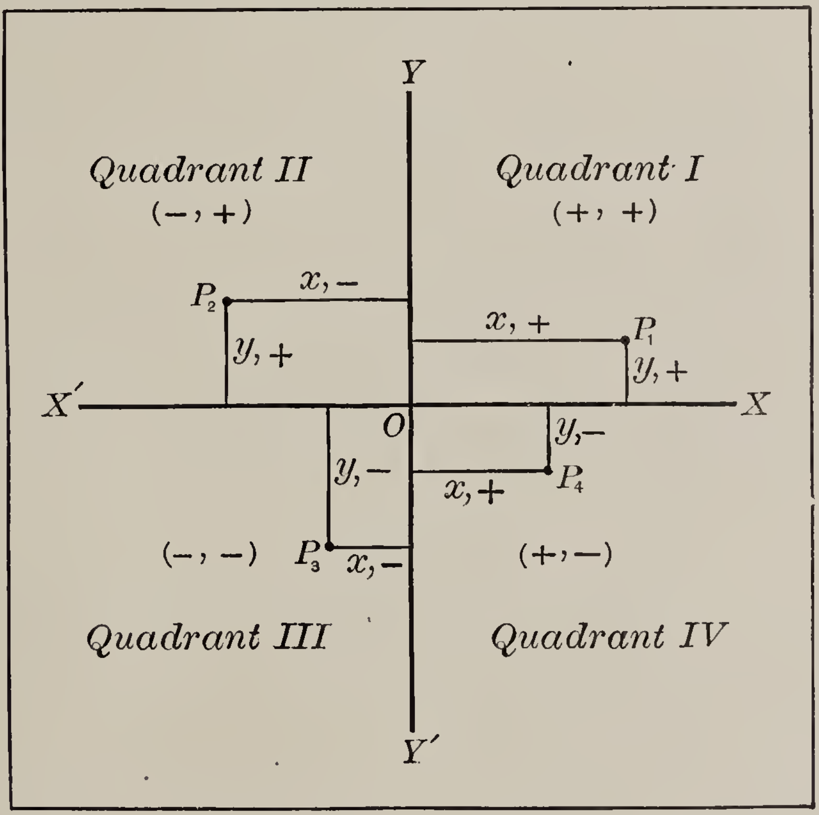
Since the point K is opposite the sixth space mark on OY and 1 space to the left of OY, it corresponds to the eleventh pair of numbers, $x = -1$, $y = 6$. Similarly R corresponds to the twelfth pair, $x = -2$, $y = 7$. This leads us to extend XO to the **left** and, dividing it into equal spaces, to number the *consecutive* points of division with **negative numbers,** -1, -2, -3, etc.

Then the line RI extended indefinitely in both directions would be the *complete* graph of the equation $x + y = 5$. Moreover, every point on this line would correspond to a pair of numerical values of x and y which satisfy this equation. These numerical values would include all the possible integers and fractions both positive and negative. The truth of this will become clearer as we proceed.

80. Definitions and assumptions. The preceding explanations and questions should tend to make clear that in constructing the graph of an equation in two variables a number of assumptions must be made. These assumptions and some necessary definitions are now stated. It is agreed:

I. To have two lines at right angles to each other, as $X'OX$, called the ***x*-axis,** and $Y'OY$, called the ***y*-axis,** as in the following figure.

II. To have a line of definite length as a unit of distance. Then the number 2 will correspond to a distance of twice the unit, the number $4\frac{1}{2}$ to a distance of $4\frac{1}{2}$ times the unit, etc.



III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the surface of the paper be the ***x*-distance** (or abscissa) of the point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the ***y*-distance** (or ordinate) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive* number, and the x-distance of a point to the *left* by a *negative* number; also the y-distance of a point *above* the x-axis be represented by a *positive* number, and the y-distance of a point *below* the x-axis by a *negative* number. Briefly, *distances measured from the axes to the* ***right*** *and* ***upward*** *are* ***positive,*** *to the* ***left*** *and* ***downward, negative.***

V. That every point in the surface of the paper corresponds to a *pair of numbers*, one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x-distance and the second the measure of the y-distance. Thus the point (2, 3) is the point whose x-distance is 2 and whose y-distance is 3.

VII. That the point of intersection of the axes be called the **origin.**

The values of the x- and the y-distances of a point are often called the **coördinates** of the point.

Though not an absolute necessity, cross-ruled paper is a great convenience in all graphical work. Excellent results, however, can be obtained with ordinary paper and a rule marked in inches and fractions of an inch for measuring distances. Hence the graphical work which follows should not be omitted because it is found inconvenient to obtain cross-ruled paper for class use.

EXERCISES

Draw two axes and locate the following points, using $\frac{1}{2}$ inch or 1 centimeter as the unit distance.

1. $(3, 5)$; $(-3, 5)$; $(-3, -5)$; $(+3, -5)$.
2. $(4, -2)$; $(-6, 4)$; $(-1, -2)$; $(+2, -4)$.
3. $(0, 4)$; $(0, -4)$; $(4, 0)$; $(-4, 0)$.
4. $(2, 2)$; $(0, 2)$; $(-2, 6)$; $(2, 0)$.
5. $(0, -5)$; $(-5, 0)$; $(0, 0)$.
6. If one coördinate of a point is zero, where is it located? Where, if both are zero?

Locating points as in the preceding exercise is called *plotting* the points.

EXERCISES

1. Find and tabulate six pairs of values of x and y which satisfy the equation $x + 2y = 8$. Draw two axes and, using $\frac{1}{2}$ inch as the unit distance, plot each of the points. Are the six points in a straight line? Where do all the points lie whose x- and y-distances satisfy the equation $x + 2y = 8$? What, then, is the graph of the equation $x + 2y = 8$? Does $x = 4$, $y = 4$, satisfy this equation? Plot the point (4, 4). Is it on the graph of the equation? If the x- and y-distances of a point satisfy the equation $x + 2y = 8$, where is the point located? If the x- and y-distances do not satisfy the equation $x + 2y = 8$, where is the point located?

Find and tabulate six pairs of values for x and y which satisfy each of the following equations. Use numbers not greater than 10. Have at least one negative value for x and one negative value for y. Then plot the six corresponding points.

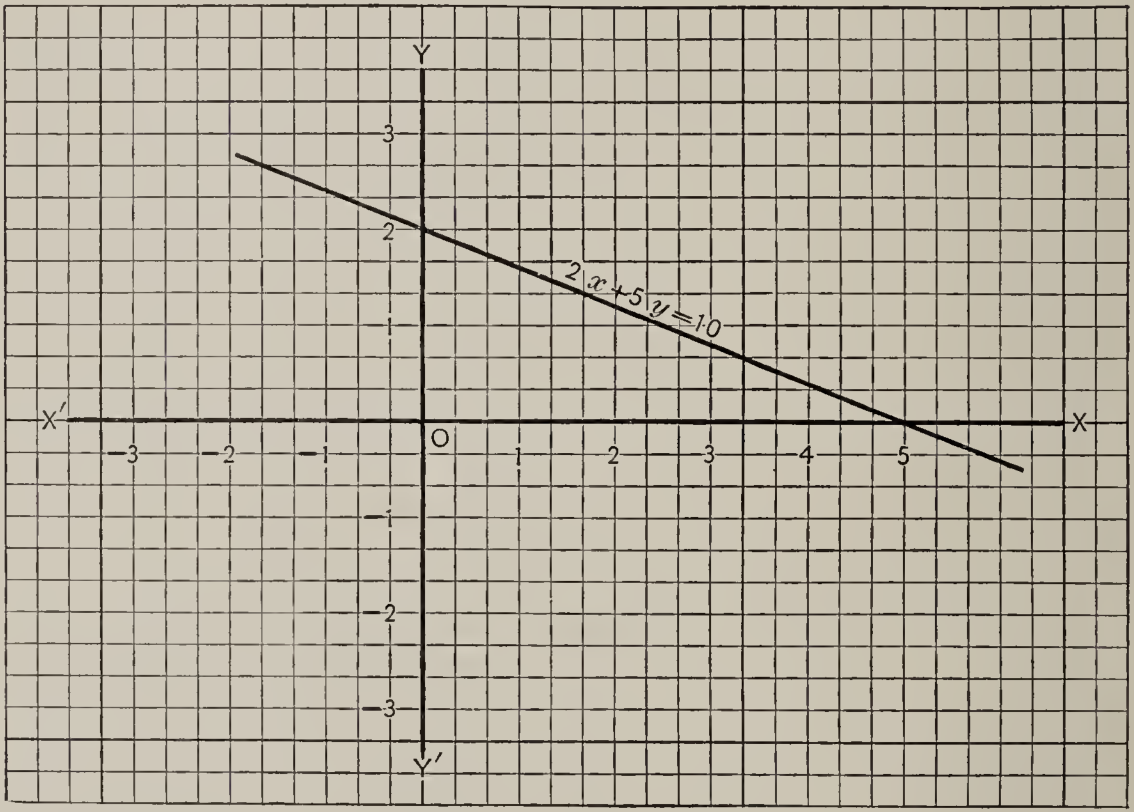
2. $3x + 2y = 6$.
4. $x + y = 0$.
6. $x = 3y$.
3. $3x - 4y = 12$.
5. $x - y = 0$.
7. $y = 2x$.

The preceding work should be enough to *convince* the student that the graph of an equation of the first degree in x and y is a *straight line.* It can be proved that the graph of any equation of the first degree (linear) in two variables is a straight line, but the student would not understand the proof were it given now. Therefore it will be *assumed* that the graph of every linear equation in two variables is a straight line. And as a straight line is determined by *any two* of its points, it will be sufficient in graphing a linear equation in two variables to plot any *two points* whose x- and y-distances satisfy the equation, and then to draw through these two points a straight line. The two points most convenient to plot are usually the two in which the line cuts the axes. Occasionally these points come very close together, and consequently they will not determine accurately the position of the line. In such cases one should decide on two values of x rather far apart (such as 0 and 5, or

0 and -5) and compute the corresponding values of y. Two such points will fix the position of the line more accurately.

If a line goes through the origin (as in Exercise 6 preceding), $x=0$, $y=0$, will do for one point, but a point outside the axes must be taken for the second one.

Example: Graph the equation $2x+5y=10$. In this equation if $x=0$, $y=2$; and if $y=0$, $x=5$. Here the point (0, 2) is on the y-axis in the adjacent figure, 2 units above the origin, and the point (5, 0) is on the x-axis, 5 units to the right of the origin. The straight line through these two points is the graph of $2x+5y=10$.



Check: If an error has been made in obtaining the value of x or y from the equation, or in plotting the values found, it can be quickly detected by plotting a third point, the values of whose x- and y-distances satisfy the equation. If this third point lies on the line determined by the first two points, the line has been correctly located; if it does not, a mistake has been made.

EXERCISES

Graph the following linear equations:

1. $x + y = 6.$

2. $x - y = 5.$

3. $x + 2y = 8.$

4. $3x + 4y = 12.$

5. $4x - 3y = 12.$

6. $2x + 4y = 9.$

7. $x - 2y = 0.$

8. $3x - y = 0.$

9. $x = 4.$

HINT. The equation $x = 4$ is equivalent to the equation $x + 0y = 4$. This last is satisfied by $x = 4$ and *any* value of y. Thus the pairs of values (4, 3); (4, 6); (4, 0); (4, – 2), etc., satisfy the equation $x + 0y = 4$. Plotting these points, it is evident that the required graph is a line parallel to the y-axis and 4 units to the right of it.

10. $x = -6.$

11. $y = 5.$

12. $y = -2.$

13. $x = 0.$

14. $y = 0.$

15. $x = \pm 3.$

16. If a point is on a line, do the values of its x- and its y-distances satisfy the equation of the line?

17. If the values of the x- and the y-distances of a point satisfy the equation of a line, is the point located on the graph of the equation?

18. Is the point (3, 4) on the line whose equation is $3x - 4y = 12$? Is (0, 4)? Is (4, 0)?

19. Can you determine without reference to the graph itself if the point (2, 6) is on any of the graphs of the equations in Exercises 1–9 above? If so, on which ones?

20. Which of the graphs of the equations in Exercises 1–15 pass through the origin?

In a linear equation containing one or more variables the *constant* term is the term which does not contain a variable.

Thus in the equation $3x + 4y = 12$, 12 is called the constant term. Also in $ax + by = c$, the constant term is c.

21. What is the value of the constant term in the equations whose graphs pass through the origin? What can be said of its value in those equations whose graphs do not pass through the origin?

22. Can you tell, then, by looking at a linear equation whether its graph goes through the origin or not? Explain.

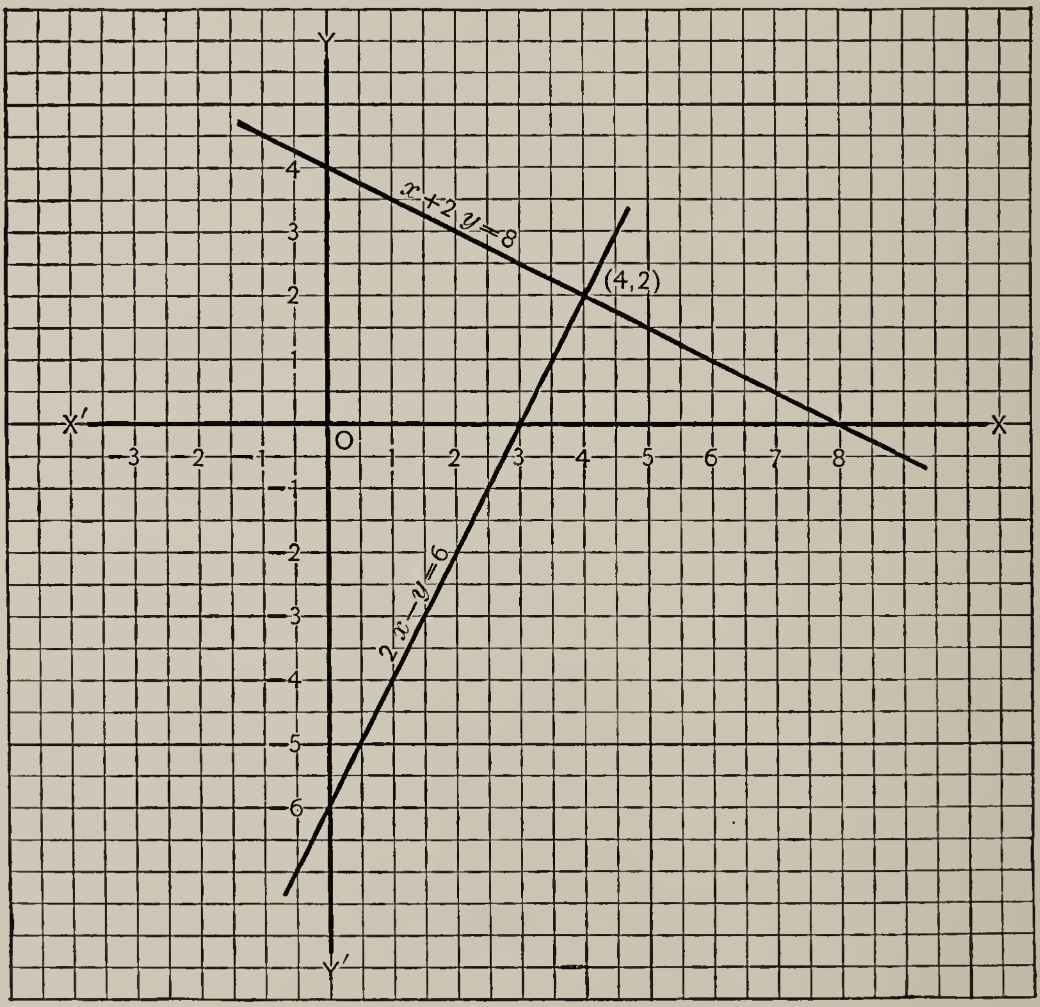
23. Can you tell from the equation when a line is parallel to the x-axis? the y-axis? Explain.

It should now be clear that:

The equation of a line is satisfied by the values of the x-distance and the y-distance of any point on that line.

Any point the values of whose x-distance and whose y-distance satisfy the equation is on the graph of the equation.

81. Graphical solution of linear equations in two variables. If we construct the graphs of the two equations $x + 2y = 8$ and $2x - y = 6$ as indicated in the adjacent figure, it is seen that for the point of intersection of the graphs x is 4 and y is 2.



Since the point (4, 2) is on both graphs, it should satisfy both equations. Substituting these values (4 for x and 2 for y) in each equation, we get the identities $4 + 4 = 8$, and $8 - 2 = 6$. Thus the **graphical solution** of two linear equations consists in plotting the two equations and finding from the graph the value of x and the value of y at the point of intersection. Since two straight lines *can intersect in but one point*, there can be but *one pair of values of x and y* which satisfies a pair of linear equations in two variables.

EXERCISES

Solve graphically the following pairs of linear equations, and verify by substituting in each pair of equations the x and y values of the point of intersection as obtained from their graphs:

1. $x + y = 6,$
   $x + 2y = 8.$
2. $x - y = 4,$
   $2x + y = 5.$
3. $x + y = 6,$
   $x - y = 4.$
4. $x + y + 8 = 0,$
   $x - y - 2 = 0.$
5. $2x + y + 2 = 0,$
   $2x + 2y + 7 = 0.$
6. $4x - y = -1,$
   $x - 2y = 5.$
7. $2x + 5y = 10,$
   $x + 4y = 8.$
8. $x + 6y = 12,$
   $2x + 3y = 12.$
9. $x + 2y = 10,$
   $x = 3y.$
10. $x + y + 6 = 0,$
    $x = y.$
11. $x + y = 4,$
    $x + y = 6.$
12. $x + 2y = 4,$
    $x + 2y = 10.$
13. $2x - y = 6,$
    $4x - 2y = 8.$
14. $x + y = 6,$
    $x - 4 = 0.$
15. $x - y = 7,$
    $y + 2 = 0.$
16. $x = 2y,$
    $x - 4 = 0.$
17. $3x - y = 0,$
    $y + 3x = 0.$
18. $x - 3 = 0,$
    $y + 2 = 0.$

Biographical note. René Descartes. One of the two or three most important advances ever made in mathematics was the discovery that algebraic equations could be represented geometrically. This great discovery was made by René Descartes (1596–1650), the French philosopher. Though never rugged in health, he took part in several campaigns when a young man, and it is said that during a weary winter spent in camp in Austria he first conceived the ideas that resulted in this important work. Though his writings read very differently from a modern book on the same subject, yet he developed all of the essentials of graphical representation. He saw that a letter, that is, a coördinate, might represent either a positive or a negative number, and so enforced upon mathematicians the conviction that negative integers are indeed numbers, and that they are useful in algebraic operations. After his time they were not usually ruled out as absurd or impossible, as was commonly the case before. He also introduced the modern exponential notation, though he did not use negative or fractional exponents. To Descartes is due the use of the last letters of the alphabet for the unknown and the first letters for the known numbers. Thus he would have written the equation $x^3 - 8x + 16 = 40$ in the form $x^3 * - 8x + 16 \infty 40$. Though the sign $=$ was used long before his time, he did not accept it. The asterisk he used to indicate that a certain power of the variable was lacking.

82. Graphical representation of statistics. Scientific data and numerical statistics from the business world are frequently exhibited with striking clearness and brevity by means of graphs. The character of the graph obtained in any case depends on the relation between the plotted numbers. Sometimes the resulting graph is a smooth curve, and then again it may be an irregular continuous line made up of straight lines of various lengths.

EXERCISES

1. A healthy man 21 years old can insure his life with a certain company for $1000 by making an annual payment of $18.40. The annual payment at a few other ages is given in the following table:

AGE	PAYMENT	AGE	PAYMENT
25	$20.14	50	$45.45
30	22.14	55	56.93
35	26.35	60	72.83
40	30.94	65	95.14
45	37.08	70	126.66

Construct a curve showing the relation between a man's age and the annual payment necessary to insure his life for $1000, as follows: Let one inch on the vertical axis correspond to

René des Cartes Philosophe
et Gentilhomme François né en Tourai
ne et decedé en Suede l'an 1650. agé de 54 ans.
Definir la Nature et de l'ame et du Corps,
De la Terre et du Ciel decouvrir les secrets,
D'un être Souverain demontrer l'existence,
Voila quel fut l'employ de René du Perron;
Et luy seul eut plus de Science
Que n'en eurent ensemble Aristote et Platon.

RENÉ DESCARTES

5 years, and one inch on the horizontal axis to $20. Then locate the points corresponding to the preceding data. Lastly, connect these points by a smooth curve.

2. In a straight line across a river the depths in feet at points 50 feet apart were as follows:

0.0
2.2
4.1
6.4
8.8
12.0
15.1
16.3
14.9
13.0
9.8
8.0
6.9
5.7
5.3
5.0
3.4
2.0
1.5
1.0
0.0

From the preceding table construct a curve showing the outline (profile) of the bed of the river at the point where the survey was made.

(Let one inch on the vertical axis equal a depth of 20 feet and one inch on the horizontal axis equal 125 feet.)

3. Starting with 100,000 persons at the age of 10, the number still living at certain ages is given in the following table:

AGE	NUMBER SURVIVING
10	100,000
15	96,285
20	92,637
25	89,032
30	85,441
35	81,822
40	78,106
45	74,173
50	69,804
55	64,563
60	57,917
65	49,341
70	38,569
75	26,237
80	14,474
85	5,485
90	847
95	3

Construct a curve showing the relation between the number of survivors and their age.

(Let one inch on the vertical axis represent 20,000 persons, and one inch on the horizontal axis represent 10 years.)

4. The public debt of the United States at five-year intervals between 1860 and 1905 is given in millions of dollars in the following table:

Year	Millions of Dollars	Year	Millions of Dollars
1860	64	1885	1872
1865	2680	1890	1549
1870	2480	1895	1717
1875	2232	1900	2132
1880	2128	1905	2293

Draw a graph of the preceding data, representing 5 years by one inch on the horizontal axis and 500 million dollars by one inch on the vertical axis.

5. The number of bushels of wheat produced in the United States each year from 1892 to 1908 is given in the following table:

Year	Millions of Bushels	Year	Millions of Bushels
1892	399	1901	522
1893	515	1902	748
1894	396	1903	670
1895	460	1904	637
1896	467	1905	552
1897	427	1906	692
1898	530	1907	735
1899	675	1908	634
1900	547		

Construct a graph from the data given.

(Let one inch on the vertical axis represent 100 million bushels, and one inch on the horizontal axis represent 4 years.)

CHAPTER XX

LINEAR SYSTEMS

83. Definitions. A **linear** equation in **one** unknown has but one root; that is, the value of the unknown in such equations is a *constant.*

A **linear** equation in **two** unknowns is satisfied by an unlimited number of pairs of values for the two unknowns. Any change in the value of one is always accompanied by a change in the value of the other. Hence the unknowns are really **variables,** as they were called in the chapter on graphs.

Two or more equations involving two or more variables are referred to as a **system** of equations.

A system of equations satisfied by the same values of the variables is called a **simultaneous** system.

A **set** of values (one for each variable) which satisfies an equation in two or more variables is sometimes called a *solution* of the equation; and a set which satisfies a system is often called a solution of the system. In this book, however, the word *solution* will be used to denote the *process of solving* either a single equation or a system. The values of the unknown which satisfy an equation in one unknown will be called *roots,* and a set of values for the variables satisfying an equation in two or more variables, or a system of such equations, will be called a **set of roots.**

An equation having one root or a limited number of roots is called a **determinate equation.**

Thus $2x = 10$ and $n^2 - 5n = -6$ are determinate equations; for the first is satisfied by $x = 5$ only, and the second is satisfied by $n = 2$, or 3 only.

A system of equations which is satisfied by one set of roots or a limited number of sets of roots is a **determinate system.**

Thus the system $\begin{cases} x + 2y = 12, \\ 4x - y = 3, \end{cases}$ is a determinate system; for it is satisfied by only one set of roots: $\begin{cases} x = 2, \\ y = 5. \end{cases}$ The system $\begin{cases} x^2 + y^2 = 25, \\ x + y = 7, \end{cases}$ is determinate also, for it is satisfied by only two sets of roots: $\begin{cases} x = 4, \\ y = 3, \end{cases}$ and $\begin{cases} x = 3, \\ y = 4. \end{cases}$

An equation which is satisfied by an unlimited number of sets of roots is called an **indeterminate equation.**

As we have seen, $x + y = 6$ is an indeterminate equation.

Also $x + y + z = 10$ is an indeterminate equation, for $x = 2$, $y = 3$, and $z = 5$ satisfy it as well as $x = 4$, $y = 5$, and $z = 1$, etc.

A system of equations which is satisfied by an unlimited number of sets of roots is called an **indeterminate system.**

Such a system as $\begin{cases} x + y + z = 8, \\ 2x - 3y - z = 9, \end{cases}$ is indeterminate, for many sets of roots which satisfy both equations can easily be found. For example, the set of roots $x = 7$, $y = 2$, and $z = -1$ satisfies the system. Another set of roots for the system is $x = 5$, $y = -1$, and $z = 4$.

84. Solution by addition or subtraction. The method of solving a system of two linear equations by addition or subtraction is illustrated in the

EXAMPLES

1. Solve the system $\begin{cases} x + 4y = 4, & (1) \\ x - 2y = 16. & (2) \end{cases}$

Solution: Eliminate x first, thus:

(1) − (2), $$6y = -12. \qquad (3)$$

(3) ÷ 6, $$y = -2. \qquad (4)$$

Substituting -2 for y in *either* (1) or (2), say (1),

$$x - 8 = 4. \qquad (5)$$

Solving (5), $$x = 12.$$

Check: Substituting 12 for x and -2 for y in (1) and (2) gives the obvious identities $\begin{cases} 12 - 8 = 4, \\ 12 + 4 = 16. \end{cases}$

2. Solve the system $\begin{cases} 13x + 3y = 14, & (1) \\ 7x - 2y = 22. & (2) \end{cases}$

Solution: Eliminate y first, as follows:

$(1) \cdot 2,$ $26x + 6y = 28.$ (3)

$(2) \cdot 3,$ $\underline{21x - 6y = 66.}$ (4)

$(3) + (4),$ $47x = 94.$ (5)

$(5) \div 47,$ $x = 2.$ (6)

Substituting 2 for x in (2), $14 - 2y = 22.$ (7)

Solving (7), $y = -4.$

Check: Substituting 2 for x and -4 for y in (1) and (2) gives

$$\begin{cases} 26 - 12 = 14, \text{ or } 14 = 14. \\ 14 + 8 = 22, \text{ or } 22 = 22. \end{cases}$$

Either x or y could have been eliminated first. The multipliers necessary to eliminate x are 7 and 13, while the multipliers necessary to eliminate y are the *more convenient numbers* 2 and 3.

When the notation $(3) - (4)$ is used in a solution, it indicates the subtraction of the first member of equation (4) from the first member of equation (3), the subtraction of the second member of equation (4) from the second member of equation (3), and the writing of the two results as equation (5). And the process of adding in a similar way the members of the two equations is indicated by writing $(3) + (4)$.

The notation $(3) \cdot 6$ indicates that both members of equation (3) are multiplied by 6, and $(3) \div 6$ indicates that both members of equation (3) are divided by 6.

With the meanings just explained it is customary to speak of the addition or the subtraction of two equations, and of the multiplication or division of an equation by a number.

The method of the preceding solutions is stated in the

RULE. *If necessary, multiply the first equation by a number and the second equation by another number, such that the coefficients of the same variable in both the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add and solve the equation thus obtained.

Substitute the value just found, in the simplest of the preceding equations which contains both variables, and solve for the other variable.

Check. Substitute for each variable in the *original* equations its value as found by the rule. If the resulting equations are not obvious identities, simplify them until they become such.

An attempt to solve by the rule the pair

$$3x - 6y = 40, \tag{1}$$
$$x - 2y = 8, \tag{2}$$

gives

$$3x - 6y = 40, \tag{3}$$
$$3x - 6y = 24. \tag{4}$$

Subtracting, $0 = 16$, an impossibility. This result indicates that (1) and (2) do not form a simultaneous system.

A system of equations like (1) and (2) is called an **incompatible** or **inconsistent system.**

The graphs of a pair of incompatible linear equations are parallel lines (see Exercises 11–13, page 199).

An attempt to solve by the rule the system $\begin{cases} x + 2y = 8, \\ 3x + 6y = 24, \end{cases}$ gives $0 = 0$. Here the second equation divided by 3 gives the first. Therefore any set of roots of the first is a set of the second. If we choose to regard the two equations as really different, which is not at all necessary, we say that they have an *infinite* (unlimited) number of sets of roots.

EXERCISES

Solve the following systems of equations and check results:

1. $x + 2y = 7,$
$5x - 2y = 11.$

2. $2x + y = 4,$
$3x - y = 21.$

3. $7m - n = 2,$
$n - 2m = -3.$

4. $10h - k = -3,$
$12h + 12k = 102.$

5. $7r - 8s = -30,$
$r + 11s = 20.$

6. $8x + y = 7,$
$11x + 2y = 28.$

7. $5l + 2p = 0,$
$3l + p = 3.$

8. $10v + 2u = 22,$
$u + 5v = 11.$

9. $12x + 5y = 14,$
$3x - 10y = 8.$

10. $2x - y = -1,$
$15x - 9y = 20.$

11. $3s - t = 12,$
$2t - 6s = 10.$

12. $5x - 3w = 2,$
$15x + 12w = -5.$

13. $x_1 - 6x_2 = 7,$
$12x_2 - x_1 = 0.$

14. $27h + 32k = 6,$
$16k - 9h = 8.$

15. $2r + 25r_1 = 15,$
$3r = 10r_1 - 44.$

16. $12n - 2m = 18,$
$3m = 18n + 10.$

85. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the

EXAMPLE

Solve the system $\begin{cases} 3x - 13y = 41, & (1) \\ 8x + 11y = 18. & (2) \end{cases}$

Solution: From (1), $3x = 13y + 41.$ (3)

Solving (3) for x in terms of y, $x = \dfrac{13y + 41}{3}.$ (4)

Substituting $\dfrac{13y+41}{3}$ for x in (2),

$$8\frac{(13y + 41)}{3} + 11y = 18. \quad (5)$$

(5) · 3, $8(13y + 41) + 33y = 54.$ (6)
Simplifying, $104y + 328 + 33y = 54.$ (7)
Collecting, $137y = -274.$ (8)
(8) ÷ 137, $y = -2.$ (9)

Substituting -2 for y in (4), $x = \dfrac{-26 + 41}{3} = 5.$

Check: Substituting 5 for x and -2 for y in (1) and (2) gives the obvious identities $15 + 26 = 41$ and $40 - 22 = 18$.

The method of the preceding solution is stated in the

RULE. *Solve either equation for one variable in terms of the other.*

Substitute this value in the equation from which it was not obtained and solve the resulting equation.

Substitute the definite value just found, in the simplest of the preceding equations which contains both variables, and solve, thus obtaining a definite value for the other variable.

CHECK. As on page 206.

EXERCISES

Solve by the method of substitution:

1. $x - 2y = 8,$
$3x + 2y = 7.$

2. $x - 2y = -12,$
$4x - y = 1.$

3. $14m - 2n = 1,$
$n - 6m = 0.$

4. $6h + 10k = 19,$
$2k = 3h.$

5. $3s + 12 = 3 + t,$
$t = s + 1.$

6. $18 + 2p = q,$
$p + q = -9.$

7. $3r + 15s = 7,$
$12 + 5s = -r.$

8. $20y - 3z = 1,$
$z - 6y = 0.$

9. $.75p + 1.5q = 3,$
$q = p - 16.$

10. $\dfrac{3x - 20}{2} = \dfrac{2x + 5y}{3},$
$10 = x - y.$

11. $\dfrac{1}{7 + 2m_1} = \dfrac{-7}{m_2 - 1},$
$m_1 = m_2 + 3.$

12. $\dfrac{5R_1 + 2R_2}{2} = 2(R_2 + 2),$
$R_1 - \frac{2}{3}R_2 = 0.$

86. Simultaneous equations containing fractions. The method of solving a system of two linear equations containing fractions is illustrated in the

EXAMPLE

Solve the system
$$\begin{cases} \dfrac{8x}{3} - \dfrac{59}{6} = \dfrac{3y}{2}, & (1) \\[2ex] \dfrac{3x}{4} = -2y - \dfrac{9}{2}. & (2) \end{cases}$$

$$\begin{array}{lrr}
\textbf{Solution: } (1)\cdot 6, & 16x - 59 = 9y. & (3) \\
\text{Transposing in (3),} & 16x - 9y = 59. & (4) \\
(2)\cdot 4, & 3x = -8y - 18. & (5) \\
\text{Transposing in (5),} & 3x + 8y = -18. & (6) \\
(4)\cdot 3, & 48x - 27y = 177. & (7) \\
(6)\cdot 16, & \underline{48x + 128y = -288.} & (8) \\
(7)-(8), & -155y = 465. & (9) \\
(9) \div -155, & y = -3. &
\end{array}$$

Substituting -3 for y in (4), $16x + 27 = 59.$

Whence $x = 2.$

Check: Substituting 2 for x and -3 for y in (1),

$$\frac{16}{3}-\frac{59}{6}=\frac{-9}{2},$$

or

$$\frac{-9}{2}=\frac{-9}{2}.$$

Substituting 2 for x and -3 for y in (2),

$$\tfrac{6}{4}=6-\tfrac{9}{2},$$

or

$$\tfrac{3}{2}=\tfrac{3}{2}.$$

As in the foregoing solution, it is usually best to clear the equations of fractions and write them in the form of (4) and (6) before attempting to eliminate one of the variables. Equations (4) and (6) are in what is called the **general form** of a linear equation in two variables. This form is represented for all such equations by $ax+by=c$. Here a, b, and c denote numbers, or known literal expressions.

EXERCISES

Solve the following systems of equations and check results:

1.
$$\begin{aligned}\frac{2x}{3}+4y&=\frac{26}{3},\\ 3x-\frac{7y}{2}&=-4.\end{aligned}$$

2.
$$\begin{aligned}m-\frac{3n}{5}&=\frac{18}{5},\\ \frac{8n}{3}+7m&=-16.\end{aligned}$$

3.
$$\begin{aligned}\frac{5R_1}{6}+\frac{R_2}{4}&=7,\\ \frac{2R_1}{3}-\frac{R_2}{8}&=3.\end{aligned}$$

4.
$$\begin{aligned}3k-\frac{12h}{5}&=18,\\ \frac{11k}{10}+\frac{17h}{2}&=53\tfrac{1}{2}.\end{aligned}$$

5.
$$\begin{aligned}\frac{2x}{9}-\frac{y}{2}&=-1,\\ x&=\frac{9y}{4}-4\tfrac{1}{2}.\end{aligned}$$

6.
$$\begin{aligned}.4x+.9y&=5.7,\\ 2x-y&=1.\end{aligned}$$

7.
$$\begin{aligned}12m&=\frac{11y}{9}+17,\\ \frac{2m}{3}-\frac{49}{12}&=\frac{5y}{12}.\end{aligned}$$

8.
$$\begin{aligned}9s&=\frac{t}{3}+\frac{23}{4},\\ \frac{7t}{3}-\frac{9}{4}&=\frac{-3s}{4}.\end{aligned}$$

9.
$$\begin{aligned}.04m+.75n&=10,\\ .8m-1.25n&=5.\end{aligned}$$

10. $$\begin{aligned}&28x-16y=56,\\&\frac{2x+7}{2}-y=4.\end{aligned}$$

11. $$\begin{aligned}&\frac{r+3}{7}-\frac{l}{5}=0,\\&5r+2=7l.\end{aligned}$$

12. $$\begin{aligned}&\frac{x+3}{5}+\frac{y+4}{10}=\frac{3}{2},\\&\frac{7x+1}{3}-\frac{11y-4}{7}=4.\end{aligned}$$

13. $$\begin{aligned}&\frac{20m+9}{7}=\frac{n}{7}+1,\\&m+\tfrac{29}{5}=n.\end{aligned}$$

14. $$\begin{aligned}&\frac{1}{x}+\frac{1}{y}=\frac{5}{6},\\&\frac{1}{x}-\frac{1}{y}=\frac{1}{6}.\end{aligned}$$

HINT. Solve Exercise 14 without clearing of fractions.

15. $$\begin{aligned}&\frac{1}{m_1}+\frac{1}{m_2}=\frac{2}{15},\\&\frac{1}{m_1}-\frac{1}{m_2}=-\frac{1}{3}.\end{aligned}$$

16. $$\begin{aligned}&\frac{6}{x}+\frac{7}{y}=\frac{3}{2},\\&\frac{7}{x}-\frac{6}{y}+5\tfrac{1}{3}=0.\end{aligned}$$

17. $$\begin{aligned}&\frac{1}{x}+\frac{1}{y}=10,\\&\frac{3}{x}+\frac{3}{y}=14.\end{aligned}$$

18. $$\begin{aligned}&\frac{5}{x}+12=17,\\&\frac{2}{x}-15y=0.\end{aligned}$$

19. $$\begin{aligned}&\frac{4r+q}{6r+q}=\frac{2}{5},\\&\frac{2r}{3}-q=5.\end{aligned}$$

20. $$\begin{aligned}&\frac{1}{t_2}+\frac{2}{t_1}=\frac{40}{t_1t_2},\\&2t_2-3t_1=0.\end{aligned}$$

21. $$\begin{aligned}&\frac{5m+3n-1}{3}=n+\frac{1}{3},\\&\frac{25m}{10m+2n}=1\end{aligned}$$

22. $$\begin{aligned}&\frac{k+5}{k+1}=\frac{l+2}{l-2},\\&\frac{11k+4}{l+k+1}=4.\end{aligned}$$

23. $$\begin{aligned}&\frac{2x_1-3x_2}{x_2-4x_1}=3,\\&\frac{8}{x_2+x_1}+\frac{8}{x_1-x_2}=0.\end{aligned}$$

24. $$\begin{aligned}&3x-2y=7,\\&\frac{2}{x-5}=-\frac{5}{y-4}.\end{aligned}$$

25. $$\begin{aligned}&\frac{2}{6+s}-\frac{4}{4-t}=0,\\&\frac{t}{4}+\frac{s}{2}+2=0.\end{aligned}$$

26. $$\frac{2}{n_1+2+n_2}=\frac{1}{5n_2+3n_1-5},\quad n_1+2n_2=4.$$

In the following problems the student should state *two* equations in *two* unknowns. Instead of using x and y, the *first letter* of the word denoting an unknown should be used to represent that unknown. Thus in Problem 6 below, n would represent the number of nickels and q the number of quarters.

The course here suggested is desirable for many reasons, and it should be followed in all problems containing two or more unknowns, unless the words denoting two of the unknowns begin with the same letter.

PROBLEMS

1. The difference of two numbers is 25 and their sum is 46. Find the numbers.

2. The quotient of two numbers is 6 and their sum is 49. Find the numbers.

3. Find two numbers whose difference is 36 and whose quotient is 3.

4. The value of a certain fraction is $\frac{3}{5}$. If 3 be added to the numerator and 1 to the denominator, the value of the resulting fraction is $\frac{3}{4}$. Find the fraction.

5. The greater of two numbers divided by the less gives a partial quotient of 3 and a remainder of 6. The less divided by the greater gives a fraction which is .7 less than 1. Find the numbers.

6. A collection of nickels and quarters, containing 77 coins, amounted to $9.85. How many coins of each kind were there?

7. If $\frac{4}{3}$ be subtracted from the numerator and $\frac{1}{9}$ added to the denominator of a certain fraction, the value of the resulting fraction is $\frac{3}{5}$. The sum of the numerator and the denominator of the original fraction is 19. Find the fraction.

8. The difference between the numerator and the denominator of a certain proper fraction is 24. If $\frac{3}{8}$ be added to the numerator and $\frac{1}{4}$ be taken from the denominator, the value of the resulting fraction is $\frac{3}{14}$. Find the fraction.

9. Two weights balance when one is 12 inches and the other 8 inches from the fulcrum. If the first weight increased by 2 pounds is placed 10 inches from the fulcrum, the balance is maintained. Find the two weights.

10. Two weights balance when one is 12 inches and the other 18 inches from the fulcrum. If the first weight is decreased 12 pounds, the other weight must be moved 3 inches nearer the fulcrum to balance. Find the weights.

11. Two weights balance when one is 15 inches and the other 10 inches from the fulcrum. The smaller weight is moved one inch nearer the fulcrum and decreased 6 pounds. Then the larger weight is decreased 12 pounds and a balance results. Find the two weights.

12. A's age is now twice B's. Seven years ago B was $\frac{1}{3}$ as old as A. Find the age of each now.

13. In 5 years A will be twice as old as B. Five years ago A was three times as old as B. Find the age of each now.

14. The perimeter of a rectangle is 232 feet and the length is 8 feet more than twice the width. Find the dimensions of the rectangle.

15. A part of \$1000 is invested at 6% and the remainder at 5%. The yearly income from both is \$54. Find the number of dollars in each investment.

16. A part of \$2000 is invested at $4\frac{1}{2}\%$ and the remainder at $3\frac{1}{2}\%$. The yearly income from the $3\frac{1}{2}\%$ investment exceeds the other yearly income by \$10. Find the number of dollars in each investment.

17. A part of \$5000 is invested at 4% and the remainder at 6%. The 4% investment yields \$126 more in 5 years than the one at 6% does in 3 years. Find the number of dollars in each investment.

18. Ten rubles are worth 10 cents less than 20 marks, and 12 marks are worth 4 rubles and a dollar. Find the value of a ruble and a mark in cents.

19. Five francs are worth 19 cents more than 2 florins, and the sum of 3 francs and one florin is worth 5 cents less than one dollar. Find the value of a florin and a franc in cents.

20. The sum of the two digits of a 2-digit number is 9. If 45 be subtracted from the number, the result will be expressed by the digits in reverse order. Find the number.

Solution: Let $t =$ the digit in tens' place,
and $u =$ the digit in units' place.
Then $t + u = 9. \qquad (1)$

But t standing in tens' place has *its numerical value multiplied by* 10. Therefore the number is represented by the binomial $10t + u$, and the number formed by the digits in reverse order is represented by the binomial $10u + t$.

Hence $10t + u - 45 = 10u + t. \qquad (2)$
Simplifying (2), $t - u = 5. \qquad (3)$
Solving (1) and (3), $t = 7,$
and $u = 2.$

Hence the number is 72.

Check: $7 + 2 = 9.$
$72 - 45 = 27.$

21. The sum of the digits of a 2-digit number is 7. If 27 be added to the number, the result is expressed by the digits in reverse order. Find the number.

22. The tens' digit of a 2-digit number is twice the units' digit. If 36 be subtracted from the number, the result is expressed by the digits in reverse order. Find the number.

23. If a 2-digit number be divided by the sum of its digits, the quotient is 4. If 36 be added to the number, the result is expressed by the digits in reverse order. Find the number.

24. If a 2-digit number be increased by 3 and then the result be divided by the sum of its digits, the quotient is 9. If the number be divided by three times the units' digit, the quotient is 17. Find the number.

25. If a 2-digit number be divided by the sum of its digits, the quotient is 7. If the number formed by the digits in reverse order be divided by the sum of the digits and 3, the quotient is 3. Find the number.

26. The sum of the reciprocals of two numbers is $\frac{5}{24}$, and the difference of their reciprocals is $\frac{1}{24}$. Find the numbers.

27. The difference of the reciprocals of two numbers is $1\frac{1}{6}$. The quotient of the greater number divided by the less is $1\frac{7}{8}$. Find the numbers.

28. If 15 grams be taken from one pan of a balance and placed in the other, the sum of the weights in the first will be $\frac{1}{2}$ the sum of those in the second. But if 85 grams be taken from the second and placed in the first, the sums of the weights in each pan will then be the same. Find the weight in each pan at first.

29. A gives B \$20; then B has twice as much money as A. B then gives A \$75 and has left $\frac{1}{5}$ as much as A. How many dollars had each at first?

30. The circumference of the fore wheel of a carriage is 2 feet less than that of the rear wheel. The fore wheel makes as many revolutions in going 155 feet as the rear wheel in going 186 feet. Find the circumference of each wheel.

31. If the length and the width of a rectangle be each increased one foot, the area will be increased 18 square feet. But if the length and the width be each decreased one foot, the area will be decreased by 16 square feet. Find the length and the breadth.

32. A and B working together can do a piece of work in $2\frac{2}{5}$ days. A works 50% more rapidly than B. How many days would each require alone?

33. A and B together can do a piece of work in $7\frac{1}{5}$ days. They work together for 5 days, and A finishes the job by himself in $3\frac{2}{3}$ days. How many days would each require alone?

34. If the length of a rectangle be increased by 4 feet and the width decreased by 2 feet, the area is increased 8 square feet. But if the length be decreased by 1 foot and the width increased by 3 feet, the area is increased 33 square feet. Find the dimensions of the rectangle in feet and its area in square yards.

35. A rectangle has the same area as one 10 feet longer and 6 feet narrower. It also has the same area as one 4 feet longer and 3 feet narrower. Find the dimensions of the rectangle.

36. The products of three pairs of numbers are equal. One number in the second pair is 2 greater, and one in the third pair 3 greater, than the first number in the first pair. The other numbers in the second and third pairs are respectively 15 less and 18 less than the second number of the first pair. Find each pair of numbers.

37. If the number of men who together purchased a piece of land had been 3 more, each would have had to pay $200 less than he did; but if the number of men had been 4 less, each would have had to pay $500 more than he did. Find the number of men and the price of the land.

38. A man rows 10 miles downstream in 2 hours and returns in 2 hours and 30 minutes. Find the rate of the river and his rate in still water.

HINT. Let $x =$ the man's rate in still water in miles an hour, and $y =$ the rate of the river in miles an hour. Then his rate downstream is $x + y$ miles an hour, and upstream $x - y$ miles an hour.

39. A boat goes downstream 36 miles in 3 hours and upstream 24 miles in 3 hours. Find its rate in still water and the rate of the current.

40. The rate of a boat in still water is $8\frac{1}{2}$ miles an hour. It goes down the river from A to B in 14 hours. It returns one half the distance from B to A in 10 hours. Find the rate of the river and the distance from B to A.

41. A boat which runs 12 miles an hour in still water goes downstream from A to C in 7 hours. It returns upstream to B, 36 miles below A, in 5 hours. Find the distance from A to C and the rate of the stream.

42. A train leaves A one hour late and runs from A to B at 25% more than its usual rate, arriving on time. If it had run from A to B at 24 miles an hour, it would have been 10 minutes late. Find the distance from A to B and the usual rate of the train.

43. A train leaves A 40 minutes late. It then runs to B at a rate 20% greater than usual, and arrives 16 minutes late. Had it run 15 miles of the distance from A to B at the usual rate and the rest of the trip at the increased rate, it would have been 22 minutes late. Find the usual rate and the distance from A to B.

44. The rate of a passenger train is 66 feet a second and the rate of a freight train 44 feet a second. When they run on parallel tracks in opposite directions they pass each other in 15 seconds. The length of the freight train is twice the length of the passenger train. Find the length of each.

45. The rate of a passenger train is 45 miles an hour and that of a freight train is 30 miles an hour. The freight train is 350 feet longer than the passenger train. When the trains run on parallel tracks in the same direction they pass each other in 1 minute and 15 seconds. Find the length of each.

46. The length of a freight train is 1540 feet and the length of a passenger train 660 feet. When they run on parallel tracks in opposite directions they pass each other in 20 seconds, and when they run in the same direction they pass each other in 1 minute and 40 seconds. Find the rates of the trains.

47. Two bicyclists travel in opposite directions around a quarter-mile track and meet every 22 seconds. When they travel in the same direction, the faster passes the slower once every 3 minutes and 40 seconds. Find the rate of each rider.

87. Literal equations in two variables. Linear systems in which the variables have literal coefficients are solved by the method of § 84.

EXERCISES

In Exercises 1–16 consider a, b, c, d, and these letters with subscripts, as known numbers; solve for the other letters involved and check. Solve Exercises 17–20 for x and y.

1. $$\begin{aligned}&3x + 7y = 17a,\\&10x - 4y = 2a.\end{aligned}$$

2. $$\begin{aligned}&3x - y = 10b,\\&4x + 9y = 3b.\end{aligned}$$

3. $$\begin{aligned}&5m - 4n = 10a - 4,\\&m - 2na = 0.\end{aligned}$$

4. $$\begin{aligned}&11h + 5k = 33c,\\&\frac{h}{c} - \frac{k}{2c} = 3.\end{aligned}$$

5. $$\begin{aligned}&12R_1 - 11R_2 = a + 12b,\\&R_1 + R_2 = 2a + b.\end{aligned}$$

6. $$\begin{aligned}&8p + 9q = 4a + 9a_1,\\&\frac{p}{2} - 3q = \frac{a - 12a_1}{4}.\end{aligned}$$

7. $$\begin{aligned}&\frac{7cx}{3} - \frac{5y}{2} = -3c,\\&x + \frac{11y}{4} = 11c + 3.\end{aligned}$$

8. $$\begin{aligned}&7.5x + 3y = 6a,\\&.25x + .5y = 0.\end{aligned}$$

9. $$\begin{aligned}&\frac{h}{2a} - \frac{2k}{a} = -5,\\&\frac{3h}{a} - \frac{7k}{4a} = \frac{3}{4}.\end{aligned}$$

10. $$\begin{aligned}&dr + 3s = 1 - d,\\&7dr + 36s = 7 - 12d.\end{aligned}$$

11. $$\begin{aligned}&h - k = 0,\\&4 - \frac{h+k}{5c} = \frac{h-k}{2c}.\end{aligned}$$

12. $$\begin{aligned}&(a+b)n = 1 - cm,\\&(a+b)m - 1 = -cn.\end{aligned}$$

13. $$\begin{aligned}&\frac{m}{b_1 - b_2} + \frac{n}{b_1 + b_2} = 2,\\&m + n = 2b_1.\end{aligned}$$

14. $$\begin{aligned}&\frac{1}{x} + \frac{1}{y} = 2b,\\&\frac{2}{x} - \frac{3}{y} = 5c - b.\end{aligned}$$

15. $$\begin{aligned}&\frac{1}{x + a_1} + \frac{1}{y} = \frac{a_1 + a_2}{2a_1a_2},\\&\frac{a_1}{x + a_1} - \frac{a_2}{y} = 0.\end{aligned}$$

16. $$\begin{aligned}&\frac{m + 2a}{n - a} = 1,\\&\frac{\frac{6}{5}n - a}{3m - 2a} - \frac{2n}{5m} = 0.\end{aligned}$$

17. $$\begin{aligned}&kx - ry = 0,\\&x + y - h = 0.\end{aligned}$$

18. $$\begin{aligned}&x + dy = 3,\\&d(x-3) - y = 0.\end{aligned}$$

19. $$\begin{aligned}&ax - by = c,\\&x + y = b.\end{aligned}$$

20. $$\begin{aligned}&ax + by = c,\\&dx + ey = f.\end{aligned}$$

GENERAL PROBLEMS

1. If one book costs a dollars, what will c books cost?

2. If a books cost b dollars, what will one book cost? c books?

3. (a) Find the perimeter and the area of a rectangle whose length is a and whose width is b. (b) Then find the perimeter and the area of a second rectangle whose dimensions are three times the first. (c) The perimeter of the second is how many times the perimeter of the first? (d) The area of the second is how many times the area of the first?

4. The base of a triangle is 8. The altitude is 10. Find the area.

5. The base of a triangle is b. The altitude is 8. Find the area.

6. The base of a triangle is b. The altitude is a. Find the area.

7. The base of a triangle is $a + b$. The altitude is $a - b$. Find the area.

8. The base of a triangle is $x - 2y$. The altitude is $x + 2y$. Find the area.

9. The area of a triangle is k. The base is b. Find the altitude.

10. The altitude of a triangle is a inches and the base is 10 inches. If 2 inches be taken from the altitude, how much must the base be increased so that the area will be the same as before?

11. The altitude of a triangle is a feet, the base is b feet. The altitude is increased h feet and the base decreased so that the area is the same as before. How many feet are taken from the base?

12. The sum of two numbers is s and their difference is d. Find the numbers.

13. The first of two numbers is a times the second, and the first minus the second is b. Find the numbers.

14. The sum of two numbers is b, and the quotient of the first divided by the second is a. Find the numbers.

15. If a be added to the numerator of a certain fraction, the value of the resulting fraction is 2. If b be added to the denominator, the value of the resulting fraction is 1. Find the fraction.

16. If the numerator of a certain fraction be increased by 1, the value of the resulting fraction is x. If the denominator of the fraction be decreased by 2, the value of the resulting fraction is y. Find the numerator and the denominator.

17. The value of a certain fraction is b. If 2 be added to the numerator, the value of the resulting fraction is c. Find the numerator and the denominator.

18. A boy who weighs a pounds and one who weighs b pounds balance at the opposite ends of a teeter board whose length is l feet. How far is the fulcrum from each end of the board?

19. A certain number of books at 80 cents each and another number at $1.10 each cost together h dollars. If the price of the books had been interchanged, the total cost would have been k dollars. Find the number of each kind.

20. Two books cost c dollars. The first cost d cents more than the second. Find the cost of each.

21. A and B have together k dollars. A gives h dollars to B and then they have equal sums. How many dollars had each at first?

22. If A gives h dollars to B, they will have equal sums. If B gives k dollars to A, A will have twice as much as B. How many dollars has each?

23. If A gives $10 to B, B will have h dollars more than A. But if B gives k dollars to A, A will have three times as much as B. How many dollars has each?

24. A and B have together \$40. A gives h dollars to B, after which B gives k dollars to A. Then they have equal sums. How many dollars had each at first?

25. A gives r dollars to B and then has $\frac{1}{2}$ as much money as B. Then B gives \$8 to A and has left $\frac{3}{4}$ as much money as A. How many dollars had each at first?

26. A part of \$1000 is invested at $a\%$ and the remainder at $b\%$. The yearly income from both investments is c dollars. How many dollars are there in each investment?

27. A portion of x dollars is invested at 5% and the remainder at 4%. The yearly income is y dollars. How many dollars are there in each investment?

28. A works three times as fast as B. Together they can do a piece of work in c days. How many days would each require alone?

29. A works h times as fast as B. Together they can do a piece of work in 4 days. How many days would each require alone?

30. A and B together can do a piece of work in h days. A can do $\frac{2}{3}$ of the work in 6 days. How many days does each require alone?

31. A and B together can do a piece of work in 5 days. A can do $\frac{2}{3}$ of it in k days. How many days does each require alone?

32. B requires twice as much time as A to do a piece of work which they can do together in n days. How many days does each require alone?

33. A and B together can do a piece of work in p days. A works q times as fast as B. How many days does each require alone?

34. A man travels n miles and then returns to his starting point. Going, his rate is 3 miles an hour; returning, it is 4 miles an hour. How many hours did the entire journey take?

35. A and B start at the same time from two towns k miles apart and travel toward each other until they meet. A travels 3 miles an hour and B travels 5 miles an hour. In how many hours do they meet? How far does each travel?

36. In Problem 34, what would the required time have been, if the rate going had been p miles an hour and the rate returning q miles an hour?

37. In Problem 35, what would have been the respective distances if A had rested h hours on the way before he met B?

38. A and B start at the same time from two points c miles apart and travel toward each other until they meet. A travels p miles an hour and B travels q miles an hour. In how many hours do they meet?

39. In Problem 38, how many miles does each travel?

40. A man rides in a carriage d miles and returns on foot at the rate of 3 miles an hour. The time of riding is h hours less than the time of walking. Find the rate of the carriage.

41. In the preceding problem, if the rate of walking had been c miles an hour, what would have been the rate of the carriage?

42. A man rides a distance of p miles and walks back at the rate of q miles an hour. The entire trip took t hours. Find his rate of riding.

43. A and B start from the same point at the same time and travel in opposite directions for n hours. They are then 50 miles apart. A travels 2 miles an hour more than B. Find the rate of each.

44. In Problem 43, what would have been the respective rates if A had traveled k miles an hour more than B, and at the end of n hours they were h miles apart?

45. A man has just t hours at his disposal. How far can he ride in a carriage which travels p miles an hour, and yet have time to walk back at the rate of q miles an hour?

88. Determinate systems in three and four variables. Consider the equations:

$$m + n + p = 6. \tag{1}$$

$$2m + 3n + 4p = 16. \tag{2}$$

$$3m + 4n + 5p = 22. \tag{3}$$

$$m + 2n + 3p = 10. \tag{4}$$

$$6m + 9n + 12p = 48. \tag{5}$$

Equation (3) is (1) plus (2); (4) is (2) minus (1); (5) is (2) multiplied by 3. Hence we speak of (3), (4), and (5), with respect to (1) and (2), as **derived** equations. Equations (1) and (2) are spoken of as **independent** with respect to each other, because neither can be derived from the other as (3), (4), and (5) were derived from (1) and (2).

A system of three *independent equations* of the first degree in three variables, no two equations being *incompatible*, has *one* set of roots and *only* one.

The method of obtaining the set of roots of a determinate system is illustrated in the following

EXAMPLE

$$\text{Solve the system} \begin{cases} m + 6n - 5p = 23, & (1) \\ 3m - 8n + 4p = -1, & (2) \\ 7m - 10n + 10p = 0. & (3) \end{cases}$$

Solution: Eliminate one variable, say p, between (1) and (2) thus:

$$\begin{array}{lrr} (1)\cdot 4, & 4m + 24n - 20p = 92. & (4) \\ (2)\cdot 5, & 15m - 40n + 20p = -5. & (5) \\ \hline (4)+(5), & 19m - 16n \qquad = 87. & (6) \end{array}$$

Now eliminate p between (2) and (3) as follows:

$$\begin{array}{lrr} (2)\cdot 5, & 15m - 40n + 20p = -5. & (7) \\ (3)\cdot 2, & 14m - 20n + 20p = 0. & (8) \\ \hline (7)-(8), & m - 20n \qquad = -5. & (9) \end{array}$$

The equations (6) and (9) contain *the same two* variables, m and n.

$$\begin{array}{lrr} (6)\cdot 1, & 19m - 16n = 87. & (10) \\ (9)\cdot 19, & 19m - 380n = -95. & (11) \\ \hline (10)-(11), & 364n = 182. & (12) \\ (12)\div 364, & n = \frac{1}{2}. & (13) \end{array}$$

Substituting $\frac{1}{2}$ for n in (9), $m - 10 = -5$. (14)

Solving (14), $m = 5$. (15)

Substituting $\frac{1}{2}$ for n and 5 for m in (1),

$$5 + 3 - 5p = 23. \qquad (16)$$

Solving (16), $p = -3$.

Check: Substituting 5 for m, $\frac{1}{2}$ for n, and -3 for p in (1), (2), and (3),

$$5 + 3 + 15 = 23, \quad \text{or} \quad 23 = 23.$$
$$15 - 4 - 12 = -1, \quad \text{or} \quad -1 = -1.$$
$$35 - 5 - 30 = 0, \quad \text{or} \quad 0 = 0.$$

For the solution of a simultaneous system of linear equations in three variables we have the

Rule. *Decide from an inspection of the coefficients which variable is most easily eliminated.*

Using any two equations, eliminate that variable.

With one of the equations just used, and the third equation, again eliminate the same variable.

The last two operations give two equations in the same two variables. Solve these two equations by the rule, pages 205–206.

Substitute the two values found in the simplest of the original equations and solve for the third variable.

Check. Substitute the values found in each of the *original* equations and simplify results.

Four or more independent equations in three variables have no common set of roots.

In general a system of $n + 1$ independent linear equations in n variables has no set of roots; a system of n independent linear equations in n variables, no two of which are incompatible, has one set of roots; and a system of $n - 1$ independent linear equations in n variables, no two of which are incompatible, has an infinite number of sets of roots.

A system of four independent equations in four variables may be solved as follows:

Use the first and second equation, then the first and third, and lastly the first and fourth, and eliminate the same variable each time. This gives a system of *three* equations in the *same three* variables, which can be solved by the rule given above.

EXERCISES

Solve the following systems:

1. $$\begin{aligned} m + n - 2p &= 13,\\ m - 3n - p &= -3,\\ m - n + 4p &= -17. \end{aligned}$$

2. $$\begin{aligned} x + y + 3z &= \tfrac{7}{2},\\ x - 2y + 4z &= 7,\\ 2x - 11y - 24z &= 5. \end{aligned}$$

3. $$\begin{aligned} x + y + z &= -1,\\ 3x - y - 5z &= 13,\\ 5x + 3y + 2z &= 1. \end{aligned}$$

4. $$\begin{aligned} 2h + 3k - 4l &= -26,\\ 3h - k + 27l &= 87\tfrac{1}{2},\\ h + 5k + 33l &= 74\tfrac{1}{2}. \end{aligned}$$

5. $$\begin{aligned} 2m + 3n - 4p &= -3,\\ m + n + 3p &= -9,\\ m + 2n - 7p &= 6. \end{aligned}$$

6. $$\begin{aligned} x + 8y + 5z &= 1,\\ 3x + 10z + 4y &= -5,\\ x + 4z &= 0. \end{aligned}$$

7. $$\begin{aligned} 2h - 3l + 4k - 2 &= 0,\\ 3h - 3l - 15 &= 0,\\ 7h - 4k - 31 &= 0. \end{aligned}$$

8. $$\begin{aligned} 4r - 10s &= 5,\\ 6r - t &= 3,\\ 5s + 2t &= -\tfrac{3}{2}. \end{aligned}$$

9. $$\begin{aligned} 2a_1 - 3a_2 &= 4,\\ 3a_1 + a_3 &= 5,\\ a_2 - 2a_3 &= 2. \end{aligned}$$

10. $$\begin{aligned} 3r_1 + 5r_2 &= 74,\\ r_1 - 2r_3 &= -16,\\ 7r_3 - 4r_2 &= 44. \end{aligned}$$

11. $$\begin{aligned} \frac{1}{m} + \frac{1}{n} - \frac{1}{p} &= 1,\\ \frac{1}{m} + \frac{1}{n} + \frac{1}{p} &= \frac{2}{3},\\ \frac{1}{m} - \frac{1}{n} + \frac{1}{p} &= 0. \end{aligned}$$

HINT. Solve Exercise 11 without clearing of fractions.

12. $$\begin{aligned} \frac{2}{m} - \frac{3}{n} + \frac{10}{p} &= -3,\\ \frac{4}{m} + \frac{5}{p} + \frac{6}{n} &= 15,\\ \frac{1}{m} - \frac{1}{n} + \frac{5}{p} &= -\frac{1}{2}. \end{aligned}$$

13. $$\begin{aligned} \frac{1}{A} + \frac{1}{B} &= 2,\\ \frac{1}{A} + \frac{1}{C} &= 3,\\ \frac{1}{B} + \frac{1}{C} &= 4. \end{aligned}$$

14. $$\begin{aligned} r + s + t + u &= 2.8,\\ r - s + t - u &= 7.2,\\ r + 2s + 3t - 5u &= 7,\\ r + s - 8t + u &= -1.7. \end{aligned}$$

15. $$\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 26,\\ \frac{4}{y} - \frac{10}{z} &= 3,\\ \frac{1}{x} + \frac{5}{z} &= \frac{25}{2}. \end{aligned}$$

Note. Perhaps the student wonders if a linear equation in three variables has a graphic representation. It may partially satisfy his curiosity to say that by means of three axes at right angles to each other such a representation, though beyond the scope of this book, is possible. Further, the points whose x, y, and z values satisfy the equation lie in a flat surface called a plane. Two such surfaces may intersect in a straight line, and the system of two equations which the surfaces represent is satisfied by the x, y, and z values of any point on this line. Three such surfaces may intersect in a single point, and the system which the surfaces represent is satisfied by the x, y, and z values of this point. In the systems of equations in three variables on page 224 the student is really finding the coördinates of the point of intersection of three planes. Those who desire more information on this subject are referred to books on analytic geometry.

Since space has but three dimensions, this method of representation of linear equations in two or three variables cannot be extended to equations containing four or more.

PROBLEMS

1. Find three numbers of which the sum of the first and second is 76, the second and third 54, and the first and third 58.

2. The sum of three numbers is 58. The sum and the quotient of two of them are 24 and 2 respectively. Find the numbers.

3. The perimeter of a triangle is 64 feet. Two of its sides are equal, and the third side is 10 feet longer than either of the first two. Find the length of each side.

4. The sum of two sides of a triangle is 52 feet and the difference is 12 feet. The perimeter of the triangle is 93 feet. Find the length of each side.

5. The sum of the two sides of a triangle which meet at one vertex is 41 feet, at another vertex 48 feet, and at the third vertex 43 feet. Find the length of each side.

6. The sum of three numbers is 26. The quotient of two of them is 9, and the sum of these two divided by the third is $3\frac{1}{3}$. Find the numbers.

Fact from Geometry. The sum of the three angles of any triangle (each angle being measured in degrees) is 180 degrees.

7. Two of the angles of a triangle are equal, and their sum is equal to the third. Find the number of degrees in each angle.

8. Two angles of a triangle are equal, and their sum is $\frac{1}{2}$ the third. How many degrees are there in each?

9. Angle A of a triangle is 17 degrees larger than angle B, and angle B is 20 degrees larger than angle C. How many degrees are there in each?

10. The sum of two angles of a triangle is 36 degrees more than the third, and the third is six times the difference of the first two. How many degrees are there in each?

11. A and B together can do a piece of work in 2 days, A and C in 3 days, and B and C in 4 days. Find the time required by each alone and by all together.

12. Two pumps together can fill a tank in 4 hours. The first of these and a third together can fill the tank in 6 hours. All three together can fill the tank in $3\frac{9}{13}$ hours. Find the number of hours required by each alone.

13. The sum of two fractions having the same denominator is 6. If 1 be added to the numerator of the first, and 1 be subtracted from the numerator of the second, the resulting fractions will be equal. If 22 be added to the denominator of each fraction, the sum of the resulting fractions is $\frac{1}{2}$. Find the fractions.

14. The sum of the digits of a 3-digit number is 15. The units' digit exceeds the tens' digit by 5. If 396 be added to the number, the result is expressed by the digits in reverse order. Find the number.

15. If the tens' and units' digits of a 3-digit number be interchanged, the resulting number is 54 greater than the number. If the tens' and hundreds' digits be interchanged, the resulting number is 360 less than the number. The sum of the digits is 10. Find the number.

16. The sum of the 4 digits of a 4-digit number is 9. The units' digit is twice the thousands' digit, and the tens' digit equals the hundreds' digit. If 2997 is added to the number, the result is expressed by the digits in reverse order. Find the number.

Fact from Geometry. The sum of the angles of any *quadrilateral* (a closed figure bounded by four straight lines) is 360 degrees.

17. Find the number of degrees in each angle of a quadrilateral in which the sum of the first and second angles is 200 degrees, the sum of the second and third 180 degrees, and the sum of the second, third, and fourth 255 degrees.

18. The sum of two opposite angles of a quadrilateral is 180 degrees and their difference is 30 degrees. The difference of the other two angles is 36 degrees. Find each angle.

19. The sum of two opposite sides of a quadrilateral is 30, the sum of the other two sides is 35, and two adjacent sides are equal. The sum of the equal sides is less by 17 than the sum of the other two. Find each side.

20. A, B, and C had together $300. A gave to B and C as many dollars as each of them had, after which B gave to A and C as many dollars as each of them then had. They then had equal amounts. How many dollars had each at first?

21. A, B, and C had together $192. A gave to B and C as many dollars as each of them had, after which B gave to A and C as many dollars as each of them then had; and, lastly, C gave to A and B as many dollars as each of them then had. They then had equal amounts. How many dollars had each at first?

CHAPTER XXI

SQUARE ROOT AND RADICALS

89. Square root of algebraic expressions. Since

$$[\pm(t+u)]^2 = t^2 + 2\,tu + u^2,$$

then the square root of

$$t^2 + 2\,tu + u^2 = \pm(t+u).$$

A study of this last form will enable us to extract the square root of any polynomial which is a perfect square. Obviously the square root of t^2 (the first term of the trinomial) is t, the first term of the root. If t^2 is subtracted from the trinomial, the remainder is $2\,tu + u^2$. The next term of the root (u) can be found by dividing the first term of the remainder ($2\,tu$) by $2\,t$, (twice the term of the root already found).

The work may be arranged thus:

$$\begin{array}{llll}
 & & \multicolumn{2}{l}{t^2 + 2\,tu + u^2 \,\underline{|\, t+u}} \\
 & & \multicolumn{2}{l}{\underline{t^2 \qquad\qquad}} \\
\text{Trial divisor,} & 2\,t & \multicolumn{1}{|l}{2\,tu + u^2} & \\
\text{Complete divisor,} & 2\,t+u & \multicolumn{1}{|l}{\underline{2\,tu + u^2 = (2\,t+u)\,u.}} &
\end{array}$$

Therefore the required roots are $\pm(t+u)$.

The foregoing process is easily extended to extracting the square root of the polynomial $4\,x^4 - 20\,x^3 + 37\,x^2 - 30\,x + 9$, whose square root contains *three* terms, as follows:

$$\begin{array}{lll}
 & & 4\,x^4 - 20\,x^3 + 37\,x^2 - 30\,x + 9 \,\underline{|\, 2\,x^2 - 5\,x + 3} \\
 & (2\,x^2)^2 = & \underline{4\,x^4 \qquad\qquad\qquad\quad} \\
\text{First trial divisor, } 2 \cdot 2\,x^2 = 4\,x^2 & & \multicolumn{1}{|l}{-20\,x^3 + 37\,x^2} \\
\text{First complete divisor, } 4\,x^2 - 5\,x & & \multicolumn{1}{|l}{\underline{-20\,x^3 + 25\,x^2 = (4\,x^2 - 5\,x)(-5\,x)}} \\
\text{Second trial divisor,} & & \\
\qquad 2\,(2\,x^2 - 5\,x) = 4\,x^2 - 10\,x & & \multicolumn{1}{|l}{12\,x^2 - 30\,x + 9} \\
\text{Second complete divisor, } 4\,x^2 - 10\,x + 3 & & \multicolumn{1}{|l}{\underline{12\,x^2 - 30\,x + 9 = (4\,x^2 - 10\,x + 3)\,3}}
\end{array}$$

Therefore the required roots are $\pm (2x^2 - 5x + 3)$.

The term $2x^2$ was obtained by taking the square root of $4x^4$; the second term, $-5x$, by dividing $-20x^3$ by the first trial divisor, $4x^2$; and the third term, 3, by dividing $12x^2$ by $4x^2$, the first term of the second trial divisor.

The method just illustrated may be stated in the

RULE. *Arrange the terms of the polynomial according to descending powers of some letter in it.*

Extract the square root of the first term. Write the result (with plus sign only) as the first term of the root, and subtract its square from the given polynomial.

Double the root already found for the first trial divisor, divide the first term of the remainder by it, and write the quotient as the second term of the root.

Annex the quotient just found to the trial divisor, making the complete divisor; multiply the complete divisor by the second term of the root, and subtract the product from the last remainder.

If terms of the polynomial still remain, double the root already found for a trial divisor, divide the first term of the trial divisor into the first term of the remainder, write the quotient as the next term of the root, form the complete divisor, and proceed as before until the process ends, or until the required number of terms of the root have been found.

Inclose the root thus found in a parenthesis preceded by the sign ±.

Note. The process of extracting the square root of numbers was familiar to mathematicians long before they knew how to find the square root of polynomials. This is consistent with the fact that the development of the methods of performing operations on literal number symbols generally followed and grew out of the similar operations on numerals. The application of the rules for extracting the square root of numbers to that of polynomials is generally ascribed to Recorde (1510–1558), who was the author of the earliest English work on algebra that we know. This book, which bears the title "The Whetstone of Wit," gives an accurate idea of the algebraical knowledge of the time, and had a very wide influence.

EXERCISES

Extract the square roots of:

1. $a^4 + 3a^2 + 2a^3 + 2a + 1$.

2. $24x^2 - 32x + 16 + x^4 - 8x^3$.

3. $21c^2 + c^4 + 20c - 10c^3 + 4$.

4. $n^6 + 9n^2 + 10n^3 + 25 - 6n^4 - 30n$.

5. $19a^2 - 11a^4 + 4a^6 - 30a + 4a^5 + 14a^3 + 25$.

6. $c^4 - 4c^3d + 6c^2d^2 - 4cd^3 + d^4$.

7. $30xy^3 + 25y^4 - 11x^2y^2 - 12x^3y + 4x^4$.

8. $-36a^4x + 36a^2x^2 + 9a^6 - 24a^3x^2 + 16x^4 + 48ax^3$.

9. $9c^4 - 2a^2b^2c^2 + 4a^3b^3c + a^4b^4 - 12abc^3$.

10. $2a^2xc^3 - 4xc^3 - 4a^2x^2 + 4x^2 + c^6 + a^4x^2$.

11. $4 - \frac{4c^2}{5} + \frac{c^4}{25}$.

12. $\frac{9}{a^2} + \frac{a^2}{4} - 3$.

13. $x^4 - 4x^3 + 5x^2 - 2x + \frac{1}{4}$.

14. $\frac{4x^2}{y^2} + 7 + \frac{y^2}{4x^2} + \frac{12x}{y} - \frac{3y}{x}$.

Solution: Arranging terms in descending powers of x and applying the rule just stated, we obtain the following:

$$\begin{array}{rl|l|l}
 & & \frac{4x^2}{y^2}+\frac{12x}{y}+7-\frac{3y}{x}+\frac{y^2}{4x^2} & \underline{\frac{2x}{y}+3-\frac{y}{2x}} \\
\left(\frac{2x}{y}\right)^2 = & & \underline{\frac{4x^2}{y^2}} & \\
2\left(\frac{2x}{y}\right) = & \frac{4x}{y} & \frac{12x}{y}+7 & \\
 & \frac{4x}{y}+3 & \underline{\frac{12x}{y}+9=\left(\frac{4x}{y}+3\right)3} & \\
2\left(\frac{2x}{y}+3\right) = & \frac{4x}{y}+6 & -2-\frac{3y}{x}+\frac{y^2}{4x^2} & \\
 & \frac{4x}{y}+6-\frac{y}{2x} & \underline{-2-\frac{3y}{x}+\frac{y^2}{4x^2}=\left(\frac{4x}{y}+6-\frac{y}{2x}\right)\left(-\frac{y}{2x}\right)} &
\end{array}$$

Therefore the square roots are $\pm\left(\frac{2x}{y}+3-\frac{y}{2x}\right)$.

15. $x^4 + 6x^3 + \frac{29x^2}{3} + 2x + \frac{1}{9}$.

16. $4a^4 + \frac{4a^3}{3} - \frac{35a^2}{9} - \frac{2a}{3} + 1$.

17. $\frac{9}{4} - \frac{127m^2}{18} + \frac{25m^4}{4} - 2m + \frac{10m^3}{3}$.

18. $9c^4 - 12c^3 + 4c^2 - \frac{4}{c} + \frac{1}{c^4} + 6$.

19. $\frac{m^2}{n^2} + \frac{n^2}{m^2} + \frac{17}{4} - \frac{5m}{n} + \frac{5n}{m}$.

20. $\frac{a^4}{25c^4} + \frac{2a^2}{c^3} + \frac{117}{5c^2} - \frac{40}{a^2c} + \frac{16}{a^4}$.

21. $\frac{a^2}{4c^4} + 9 + \frac{4c^2}{25a^4} - \frac{3a}{c^2} + \frac{2}{5ac} - \frac{12c}{5a^2}$.

22. Extract the fourth root of the expressions in Exercises 2 and 6 on the preceding page.

HINT. The fourth root of a number equals the square root of its square root.

90. Square root of arithmetical numbers. Since $1 = 1^2$, and $81 = 9^2$, a 1-digit or a 2-digit square has only *one* digit in its square root.

And as $100 = 10^2$, and $9801 = (99)^2$, a 3-digit or a 4-digit square has *two* digits in its square root.

Also $10{,}000 = 100^2$, and $998{,}001 = (999)^2$; hence a 5-digit or a 6-digit square has *three* digits in its square root.

The preceding examples illustrate the relation between the number of digits in a number and the number of digits in its square root. They also suggest a method of obtaining the first digit in the square root of any number.

For example, take the four numbers 78′43′56, 7′84′35, .98′01, and .03′27′40, and beginning at the decimal point in each number, point off periods of two digits each, as indicated. Any incomplete period on the right, as in .03′27′4, should be completed by annexing one zero; thus, .03′27′40. Now the first digit in the square root is the greatest integer whose square is less than or equal to the left-hand period. This is true whether the latter contains *two* digits or *one*. Therefore the first digit in the square root of 78′43′56 is 8, in the square root of 7′84′35 is 2, in the square root of .98′01 is 9, and in the square root of .03′27′40 is 1.

Moreover the number of digits in the square root of a perfect square is equal to the number of periods, provided any *single digit* remaining on the left is counted as a period.

Just how t and u are involved in the square of $(t+u)$, or $t^2+2tu+u^2$, is obvious on inspection, because the parts t^2, $2tu$, and u^2 cannot be united into one term. In the square of an arithmetical number, however, the parts are united. Thus $(53)^2=(50+3)^2=2500+300+9=2809$. Now it is clear how 50 and 3 are involved in $2500+300+9$, but it is not plain from 2809 alone. Pointing off, however, enables us to discover at once the first digit, 5, which is equivalent to 5 tens, or 50. With the exception of pointing off, the method of extracting the square root of an arithmetical number does not differ greatly from the method of extracting the square root of an algebraic expression. In fact, the formula, the square root of $t^2+2tu+u^2=\pm(t+u)$, can be used to explain the two processes.

If t denotes the tens and u the units, $t^2+2tu+u^2$ is closely related to $2500+300+9$, t^2 being 2500, or $(50)^2$; u^2 being 9, or 3^2; and $2tu$ being $2\cdot 50\cdot 3$. Therefore the process of extracting the square root of 2809 may be based on these relations and the work arranged as follows:

$$
\begin{array}{rl}
 & 2809\,\underline{|\,50+3} \\
t^2 = & 2500 \\
2t=2\cdot 50=100 & \overline{|\,309} \\
2t+u=100+3 & |\,\underline{309=(100+3)\,3=(2t+u)\,u=2tu+u^2}
\end{array}
$$

Therefore ± 53 are the two square roots of 2809.

If the number has three digits in its square root, the work and explanations may be arranged thus:

$$\begin{array}{ll}
 & 1'74'24\,\underline{|\,100+30+2} \\
t^2 = 10{,}000 & \underline{10000} = 10 \text{ tens squared} \\
\text{First trial divisor,} & \\
\qquad 2t = 2\cdot 100 = 200 & \;7424 \\
\text{First complete divisor,} & \\
\qquad 2t+u = 200+30 = 230 & \;\underline{6900} = (2\cdot 10 \text{ tens} + 30 \text{ units})\,30 \\
\text{Second trial divisor,} & \\
\qquad 2t = 2\cdot 130 = 260 & \;\;524 \\
\text{Second complete divisor,} & \\
\qquad 2t+u = 260+2 = 262 & \;\;\underline{524} = (2\cdot 13 \text{ tens} + 2 \text{ units})\,2
\end{array}$$

Therefore ±132 are the square roots of 17,424.

When the method and reasons for the process have become familiar, the work may be shortened by omitting the explanations and unnecessary zeros as follows:

$$\begin{array}{r|l}
\multicolumn{2}{l}{28'09\,\underline{|\,53}} \\
\multicolumn{2}{l}{25} \\
103 & 309 \\
 & \underline{309}
\end{array}
\qquad\qquad
\begin{array}{r|l}
\multicolumn{2}{l}{1'74'24\,\underline{|\,132}} \\
\multicolumn{2}{l}{1} \\
23 & 74 \\
 & \underline{69} \\
262 & 524 \\
 & \underline{524}
\end{array}$$

The method just illustrated for extracting the positive square root of a number is the one commonly used. For it we have the

RULE. *Begin at the decimal point and point off as many periods of two digits each as possible: to the left if the number is an integer, to the right if it is a decimal; to both the left and the right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand period, and write this integer for the first digit of the root.

Square the first digit of the root, subtract its square from the first period, and annex the second period to the remainder.

Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next period for a new dividend.

Double the part of the root already found for a new trial divisor and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal period in the number.

Check. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

Sometimes in using a trial divisor we obtain too great a quotient for the next digit of the root. This happens in obtaining the second digit of the square root of 32,301, where 2 into 22 gives 11. Obviously 10 and 11 are both impossible. If 9 is tried, we get $9 \cdot 29$, or 261, which is greater than 223. Similarly 8 is too great. But $7 \cdot 27 = 189$, which is less than 223. Therefore 7 is the second digit of the root.

```
  3′23′01|1
  1
2 |223
```

With practice, in cases like the one just explained, the student will be able to look ahead and decide mentally on the proper digit of the root.

Occasionally the trial divisor gives a quotient less than 1. This indicates that the required root digit is 0, which should be written in the root. The next period should then be brought down. An instance of this kind occurs in finding the second digit in the square root of 9′42.49. The quotient of $4 \div 6$ is $\frac{2}{3}$, which is not an integer. Therefore the second digit of the root is less than 1. Then the next period, 49, should be brought down. The new trial divisor will be 60, which will give 7 as the third digit of the root. The work can easily be completed, giving 30.7 as the square root.

```
  9′42.49|3
  9
6 |  42
```

An attempt to extract the square root of 2 by annexing decimal periods of zeros and applying the rule becomes a never-ending process:

The number 2 has no exact square root, and no matter how far the work be carried, there is no final digit. As the work stands, we know that the square root of 2 lies between 1.414 and 1.415. It is correct to say that 1.414 is approximately the square root of 2, or that it is the square root of 2 to three decimal places. If a closer approximation is desired, it can be obtained by extracting the square root to four or more decimal places.

$$\begin{array}{r|l}
 & 2.00'00'00\ \underline{|\,1.414} \\
 & 1 \\ \hline
24 & 100 \\
 & 96 \\ \hline
281 & 400 \\
 & 281 \\ \hline
2824 & 11900 \\
 & 11296 \\ \hline
\end{array}$$

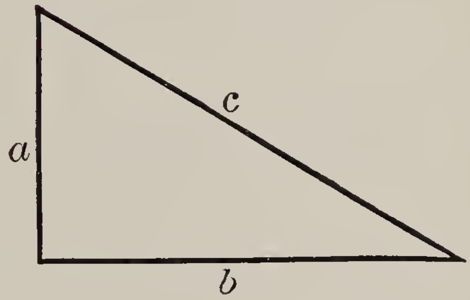
A common fraction, or the fractional part of a mixed number, should be reduced to a decimal before extracting the square root, unless the root is seen to be exact.

EXERCISES

Extract the square root, correct to three decimal places, of:

1. 6241.
2. 16129.
3. 223,729.
4. 2.
5. 5.
6. 7.135.
7. .6279.
8. .0451.
9. .0035.
10. $\frac{11}{5}$.
11. $1\frac{6}{7}$.
12. $\frac{2}{21}$.

Fact from Geometry. In the adjacent right triangle $a^2 + b^2 = c^2$; a and b are called the **legs**; and c, the side opposite the right angle, is called the **hypotenuse.**



If leg a is 8 and leg b is 15, then substituting in $a^2 + b^2 = c^2$ gives $64 + 225 = c^2$. Whence $289 = c^2$, and $c = \pm 17$.

Since -17 is not a practical answer, it is rejected.

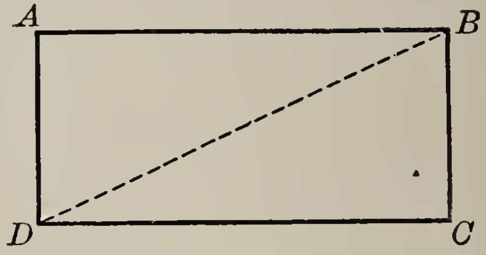
Find the hypotenuse and the area of a right triangle whose legs are:

13. 84 and 13.
14. 133 and 156.
15. 645 and 812.

Find the other leg and the area of a right triangle in which the hypotenuse and one leg are respectively:

16. 65 and 56. **17.** 397 and 325.

In rectangle $ABCD$, line DB is called a **diagonal.**



18. Find the diagonal of a rectangle whose adjacent sides are 24 feet and 143 feet.

19. One diagonal of a rectangle is 401 and one side is 399. Find the other side and the area.

20. One diagonal of a rectangle is 677 and one side is 52. Find the perimeter of the rectangle.

21. A rectangle is 7 yards longer than it is wide. Its perimeter is 102 feet. Find one diagonal.

22. One diagonal of a square is 74 meters. Find the side.

23. The side of a square is 52 inches. Find one diagonal.

24. A rectangle is 2.4 times as long as it is wide. One diagonal is 52. Find the length and the width.

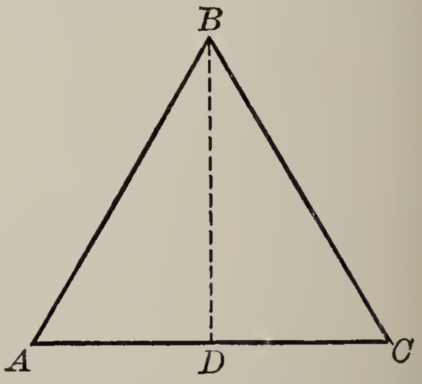
25. The width of a rectangle is 25% less than the length. The diagonal is 100. Find the area.

26. The length of a rectangle is 10. The diagonal is twice the shorter side. Find the width.

Fact from Geometry. A line drawn from one vertex of an equilateral triangle to the middle point of the opposite side is perpendicular to it.

Then in the equilateral triangle ABC, if D is the middle point of AC, BD is the altitude; and

$$\overline{BD}^2 = \overline{AB}^2 - \overline{AD}^2 = \overline{AB}^2 - \left(\frac{AC}{2}\right)^2.$$



27. If BC in the adjacent triangle is 6, find BD and the area of the triangle.

28. If AC is 10, find BD and the area of the triangle.

29. If BD is 10, find AB and the area of the triangle.

30. The perimeter of an equilateral triangle is 36. Find the altitude.

31. The altitude of an equilateral triangle is 25 centimeters. Find one side.

Note. A method of extracting the square root of numbers not unlike that in use to-day was employed by the Greek, Theon, about 350 A.D. In the Middle Ages square roots were extracted with a fair degree of accuracy by using the formulas of approximation:

$$(1)\ \sqrt{a^2+x}=a+\frac{x}{2a}\cdot \qquad (2)\ \sqrt{a^2+x}=a+\frac{x}{2a+1}\cdot$$

The true value of the square root of the number was proved to be between the results obtained by these expressions. Thus if $\sqrt{65}$ was desired, it was noticed that $65 = 64 + 1$, and from (1)

$$\sqrt{65}=\sqrt{64+1}=\sqrt{8^2+1}=8+\frac{1}{2\cdot 8}=8\tfrac{1}{16},$$

while from (2)

$$\sqrt{65}=\sqrt{64+1}=\sqrt{8^2+1}=8+\frac{1}{2\cdot 8+1}=8\tfrac{1}{17}.$$

Thus the true value of $\sqrt{65}$ is between these two numbers. This method was known to the Arabs.

It should be kept in mind that the use of decimal fractions and of the decimal point was not common until the eighteenth century. Consequently the complete application of the method of extracting the square root given in the text is comparatively recent.

91. Radicals. All the numbers of algebra are in one or the other of two classes — **real** numbers and **imaginary** numbers.

Thus 3, -5, $\sqrt{2}$, $\frac{2}{3}$, and 1.763 are real numbers.

Real numbers are of two kinds — **rational** numbers and **irrational** numbers.

A **rational** number is a positive or a negative *integer* or a number which may be expressed as the *quotient of two such integers.*

Thus 7, $\frac{3}{5}$, 4.237 are rational numbers.

A rational number can be obtained from the number 1 by carrying out the operations of addition, subtraction, multiplication, and division, which are therefore called rational operations.

Any real number which is not a rational number is an **irrational** number.

Thus $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[4]{7}$, are all irrational numbers, and cannot be expressed as the quotient of two integers.*

The $\sqrt{2}$ to six places is 1.414213. And it can be proved that the digits in the decimal portion never repeat themselves in groups of digits which have a definite order, however far the process of extracting the root be carried. Hence the decimal portion of the root is said to be non-repeating. For example, .121212 $\cdots$ is a repeating decimal. As a never-ending decimal which does not repeat cannot be expressed as the quotient of two integers, the $\sqrt{2}$ is an irrational number. The $\sqrt[3]{2}$, $\sqrt[5]{4}$, etc., are also irrational numbers. It is beyond the scope of this book, however, to show how their approximate values are obtained.

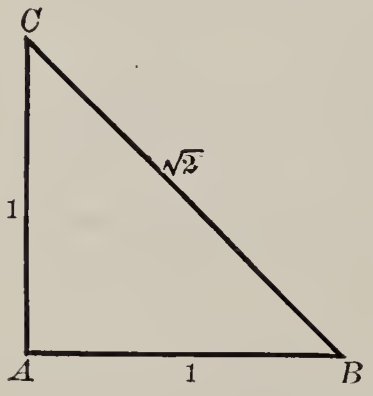
Symbols like $\sqrt{-1}$, $\sqrt{-4}$, $\sqrt[4]{-16}$, were mentioned on page 102. Such symbols arise when we express an even root of a negative number. These indicated roots are called **imaginary** numbers and will be treated later.

A **radical** is an indicated root of any algebraic expression.

Thus $\sqrt{4}$, $\sqrt{3}$, $\sqrt[3]{a}$, and $\sqrt{x^2 - 5x + 6}$ are *radicals.*

A **surd** is an irrational root of a rational number. Surds are always irrational numbers.

Thus $\sqrt{3}$, $\sqrt[3]{7}$, etc., are *surds.*



Though no irrational numbers can be expressed exactly in decimals, we can represent a few surds by the lengths of lines. Thus in the right triangle ABC, if $AB = AC = 1$ inch, $BC = \sqrt{2}$ inches. If AB were 2 inches and AC were 1 inch, BC would be $\sqrt{5}$ inches.

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* See Hawkes's "Advanced Algebra," page 52.

An irrational number is not necessarily a surd. The length of the circumference of a circle divided by the length of its diameter gives a number which is not rational. The symbol for this number is the Greek letter π (pronounced pī). The approximate value of π is $\frac{22}{7}$; more closely it is 3.1416. The number which π represents is a never-ending, non-repeating decimal whose value correct to ten decimals is 3.1415926535.

Strictly speaking, the $\sqrt{\pi}$ is not a surd, nor is an expression like $\sqrt{\sqrt{3}+2}$ a surd. The $\sqrt{\sqrt{3}}$ is a surd, for it can be written $\sqrt[4]{3}$, as we shall see later.

The **index** of a radical is the *numerical*, or *literal*, part of the radical sign.

The index determines the **order** of the radical and indicates the root to be extracted.

In $5\sqrt[3]{7}$, 3 is the index, the radical is of the third order, and the coefficient is 5.

The **radicand** is the number, or expression, under the radical sign.

In $\sqrt{9}$ and $\sqrt[3]{ax}$, 9 and ax are the radicands.

For a given index the **principal root** of a number is its *one real root*, if it has but one; or its *real positive root*, if it has two real roots.

The principal root of $\sqrt[3]{-27}$ is -3. That of $\sqrt[4]{16}$ is $+2$, not -2.

Radical expressions may be written in two ways, with **radical signs** or with **fractional exponents.** The relation between the two will now be explained. To do this it is necessary to extend the meaning of the term *exponent*, which, as defined on page 7, applied to integral exponents only. We shall assume that the laws which govern integral exponents hold for fractional exponents also.

The fact that $x^2 \cdot x^3 = x^5$, illustrates the more general law, $x^a \cdot x^b = x^{a+b}$, where a and b represent either integers or fractions.

Accordingly $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1$ or x. Since $x^{\frac{1}{2}}$ multiplied by itself gives x, *$x^{\frac{1}{2}}$ must be another way of writing the square root of x.*

Hence $\sqrt{x}$ may be written $x^{\frac{1}{2}}$.

Then $4^{\frac{1}{2}} = \sqrt{4} = 2,$

and $(25\,a^2)^{\frac{1}{2}} = \sqrt{25\,a^2} = 5\,a.$

Further, $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^1 = x.$

And since $x^{\frac{1}{3}}$ is one of the three equal numbers whose product is x, $x^{\frac{1}{3}}$ *is another way of writing the cube root of* x.

Therefore $\sqrt[3]{x}$ may be written $x^{\frac{1}{3}}$.

This means that $8^{\frac{1}{3}} = \sqrt[3]{8} = 2,$

and $64^{\frac{1}{3}} = 4.$

Similarly $\sqrt[4]{x} = x^{\frac{1}{4}},$

and $\sqrt[5]{x} = x^{\frac{1}{5}}$, etc.

In general terms, $\boldsymbol{\sqrt[n]{x} = x^{\frac{1}{n}}}.$

Now $x^{\frac{3}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = \left(x^{\frac{1}{2}}\right)^3 = \left(\sqrt{x}\right)^3,$

and $x^{\frac{3}{2}} = x^{3 \cdot \frac{1}{2}} = (x^3)^{\frac{1}{2}} = \sqrt{x^3}.$

Hence $\left(\sqrt{x}\right)^3 = \sqrt{x^3},$

and $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = \left(\sqrt{4}\right)^3$, or $\sqrt{4^3}$

for both $\left(\sqrt{4}\right)^3$ and $\sqrt{4^3}$ equal 8.

In like manner, $x^{\frac{2}{3}} = x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = \left(x^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{x}\right)^2.$

But also $\sqrt[3]{x^2} = (x^2)^{\frac{1}{3}} = (x \cdot x)^{\frac{1}{3}} = x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{2}{3}}.$

Therefore $\left(\sqrt[3]{x}\right)^2 = \sqrt[3]{x^2}.$

Again, $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2,$

or $\sqrt[3]{8^2} = 2^2$ or 4.

That is, $\left(\sqrt[3]{8}\right)^2 = \sqrt[3]{8^2}.$

In general terms, $\boldsymbol{x^{\frac{a}{n}} = \sqrt[n]{x^a}},$

or, in words, x with an exponent $\frac{a}{n}$ means the nth root of x to the ath power.

The student should fix in mind that the *denominator* of the fractional exponent is the *index* of the root, and the *numerator* the *power* to which the radicand is raised.

EXERCISES

Write with radical signs:

1. $x^{\frac{4}{3}}$.
2. $x^{\frac{5}{2}}$.
3. $(cd)^{\frac{3}{2}}$.
4. $(2x)^{\frac{1}{3}}$.
5. $2x^{\frac{1}{3}}$.
6. $5a^{\frac{3}{4}}$.
7. $(5a)^{\frac{3}{4}}$.
8 $3rx^{\frac{2}{3}}$.
9. $h^{\frac{2}{3}}k^{\frac{2}{3}}$.
10. $7s^{\frac{2}{3}}(t+w)^{\frac{1}{2}}$.
11. $3a^{\frac{5}{2}}(bc)^{\frac{1}{3}}$.
12. $5^{\frac{1}{2}}x^{\frac{2}{3}}k^{\frac{5}{4}}$.
13. $4^{\frac{1}{3}}t^{\frac{1}{a}}$.
14. $2x^{\frac{1}{b}}y^{\frac{c}{d}}$.

Find the numerical values of:

15. $25^{\frac{1}{2}}$.
16. $27^{\frac{1}{3}}$.
17. $16^{\frac{1}{4}}$.
18. $4^{\frac{3}{2}}$.
19. $64^{\frac{2}{3}}$.
20. $125^{\frac{2}{3}}$.
21. $(-8)^{\frac{4}{3}}$.
22. $32^{\frac{1}{5}}$.
23. $81^{\frac{3}{4}}$.
24. $(-216)^{\frac{2}{3}}$.
25. $(\frac{1}{16})^{\frac{1}{2}}$.
26. $(\frac{1}{25})^{\frac{1}{2}}$.
27. $(\frac{1}{16})^{\frac{1}{4}}$.
28. $25^{\frac{1}{2}}\cdot 4^{\frac{5}{2}}$.
29. $4^{\frac{1}{2}}\cdot(\frac{1}{4})^{\frac{1}{2}}$.
30. $(-32)^{\frac{4}{5}}(-64)^{\frac{1}{3}}$.
31. $36^{\frac{1}{2}}\cdot(\frac{1}{9})^{\frac{1}{2}}$.
32. $9^{\frac{1}{2}}\cdot(\frac{1}{27})^{\frac{1}{3}}$.
33. $2(\frac{4}{9})^{\frac{1}{2}}\cdot(\frac{1}{8})^{\frac{1}{3}}$.
34. $\sqrt[5]{32}\cdot 4^{\frac{1}{2}}$.
35. $\sqrt[3]{8^2}\cdot 25^{\frac{3}{2}}$.
36. $121^{\frac{1}{2}}\cdot\sqrt{\frac{25}{121}}$.
37. $(-343)^{\frac{1}{3}}\cdot\sqrt{\frac{1}{49}}$.
38. $\frac{2}{3}(\frac{169}{144})^{\frac{1}{2}}\div(\frac{1}{216})^{\frac{1}{3}}$.

Write with fractional exponents and simplify results:

39. $\sqrt[2]{a^3}$.
40. $\sqrt[2]{ax^4}$.
41. $3\sqrt{2x^5}$.
42. $\sqrt{9x}$.
43. $5\sqrt{16ax^2}$.
44. $\sqrt[3]{a^2}$.
45. $\sqrt[3]{ax^4}$.
46. $2\sqrt[3]{2x^2}$.
47. $3\sqrt[3]{8x^4}$.
48. $4\sqrt[3]{27ax^3}$.
49. $2\sqrt[4]{a^2x^3}$.
50. $4\sqrt[4]{16x}$.
51. $7\sqrt[5]{rs^7}$.
52. $5a\sqrt[5]{32m^2}$.
53. $12x^2\sqrt[3]{ax^3}$.
54. $c\sqrt[2]{(de)^3}$.
55. $uv\sqrt[2]{(u+v)^5}$.
56. $3\sqrt[2]{a^3}\cdot\sqrt[3]{x^2}$.
57. $2a\sqrt[5]{ax^2}\cdot\sqrt[3]{2m}$.
58. $\sqrt[n]{x^a}\cdot\sqrt[n]{y^b}$.
59. $\sqrt[n]{x^a}\cdot\sqrt[n]{x^{3a}}$.

92. Simplification of radicals. The form of a radical expression may be changed without altering its numerical value.

For example, $\frac{1}{\sqrt{2}}$ can be changed to $\frac{\sqrt{2}}{2}$, for each equals .707+.

Study the following changes of form:

1. $\sqrt{36} = \sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6.$
2. Similarly $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$
3. More generally, $\sqrt[2]{a^2b} = \sqrt{a^2}\sqrt{b} = a\sqrt{b}.$
4. $\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8}\sqrt[3]{3} = 2\sqrt[3]{3}.$
5. More generally, $\sqrt[3]{a^3b} = \sqrt[3]{a^3}\sqrt[3]{b} = a\sqrt[3]{b}.$
6. Finally, $\boldsymbol{\sqrt[n]{a^nb} = \sqrt[n]{a^n}\sqrt[n]{b} = a\sqrt[n]{b}}.$

Note. Although the Arabs were by no means able to state all the rules explained in this chapter, it is interesting to note that they did recognize the truth of a few of them. For instance, a writer about 830 A.D. gives, in his own notation, of course, the facts contained in the formulas $a\sqrt{b} = \sqrt{a^2b}$, and $\sqrt{a}\sqrt{b} = \sqrt{ab}$.

A radical is in its **simplest form** when the radicand:

(*a*) *Is integral.*

(*b*) *Contains no rational factor raised to a power which is equal to, or greater than, the order of the radical.*

(*c*) *Is not raised to a power, unless the exponent of the power and the index of the root are prime to each other.*

For the meaning of (*a*), (*b*), and (*c*) study carefully the

EXAMPLES

Of (*a*): 1. $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{1}{4} \cdot 6} = \sqrt{\frac{1}{4}}\sqrt{6} = \frac{1}{2}\sqrt{6}.$

2. $6\sqrt{\frac{1}{3}} = 6\sqrt{\frac{3}{9}} = 6\sqrt{\frac{1}{9} \cdot 3} = 6 \cdot \frac{1}{3}\sqrt{3} = 2\sqrt{3}.$

3. $\sqrt[3]{\frac{3}{16}} = \sqrt[3]{\frac{12}{64}} = \sqrt[3]{\frac{1}{64} \cdot 12} = \sqrt[3]{\frac{1}{64}}\sqrt[3]{12} = \frac{1}{4}\sqrt[3]{12}.$

4. $\sqrt{\frac{3}{5x}} = \sqrt{\frac{15x}{25x^2}} = \sqrt{\frac{1}{25x^2} \cdot 15x} = \frac{1}{5x}\sqrt{15x}.$

Of (*b*): 1. $\sqrt{4x^5} = \sqrt{4x^4 \cdot x} = \sqrt{(2x^2)^2 \cdot x} = 2x^2\sqrt{x}.$

2. $5\sqrt[3]{24x^5} = 5\sqrt[3]{8x^3 \cdot 3x^2} = 5\sqrt[3]{(2x)^3 \cdot 3x^2} = 10x\sqrt[3]{3x^2}.$

3. $\sqrt{16 - 8\sqrt{2}} = \sqrt{4(4 - 2\sqrt{2})} = 2\sqrt{4 - 2\sqrt{2}}.$

Of (*c*): 1. $\sqrt[4]{4} = \sqrt[4]{2^2} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$

2. $\sqrt[6]{9} = \sqrt[6]{3^2} = 3^{\frac{2}{6}} = 3^{\frac{1}{3}} = \sqrt[3]{3}.$

3. $\sqrt[4]{a^2b^4} = a^{\frac{2}{4}}b^{\frac{4}{4}} = a^{\frac{1}{2}}b = b\sqrt{a}.$

EXERCISES

Express in simplest form:

1. $\sqrt{12}$.

2. $\sqrt{32}$.

3. $\sqrt{75}$.

4. $2\sqrt{50}$.

5. $\sqrt[3]{40}$.

6. $\sqrt[3]{54}$.

7. $\sqrt{20}$.

8. $\sqrt[3]{3000}$.

9. $2\sqrt[3]{24}$.

10. $\sqrt{\frac{1}{2}}$.

11. $\sqrt{\frac{2}{5}}$.

12. $\sqrt{\frac{3}{8}}$.

13. $3\sqrt{\frac{9}{2}}$.

14. $5\sqrt{\frac{3}{5}}$.

15. $\sqrt[3]{\frac{1}{2}}$.

16. $4\sqrt[3]{\frac{1}{4}}$.

17. $8\sqrt[3]{\frac{7}{4}}$.

18. $\sqrt{1-(\frac{1}{2})^2}$.

19. $\sqrt{1+(\frac{1}{3})^2}$.

20. $\sqrt{2+(\frac{3}{5})^2}$.

21. $\sqrt[3]{1-(\frac{1}{2})^2}$.

22. $\sqrt[3]{4+(\frac{2}{3})^2}$.

23. $\sqrt[3]{1+(\frac{1}{3})^3}$.

24. $\sqrt[4]{25\,a^2}$.

25. $3\sqrt{4\,x^5}$.

26. $a\sqrt[6]{8\,x^3}$.

27. $x\sqrt[4]{49\,a^2x^2}$.

28. $\sqrt[6]{125\,x^{3a}y^{3m}}$.

29. $\sqrt{\dfrac{a^3}{27}}$.

30. $a\sqrt[3]{\dfrac{5}{2\,a^2}}$.

31. $\sqrt{R^2-\left(\dfrac{R}{2}\right)^2}$.

32. $\sqrt{R^2+\left(\dfrac{R}{2}\right)^2}$.

33. $\sqrt{\left(a+\dfrac{a}{3}\right)\left(a-\dfrac{a}{3}\right)}$.

34. $\sqrt[3]{R^3-\left(\dfrac{R}{2}\right)^3}$.

35. $(x-y)\sqrt{\dfrac{x+y}{x-y}}$.

36. $\sqrt{4+4\sqrt{2}}$.

37. $\sqrt{8-4\sqrt{2}}$.

38. $\sqrt{18+9\sqrt{3}}$.

39. $\sqrt{25\sqrt{5}-100}$.

40. $\sqrt[3]{16+8\sqrt{2}}$.

41. $\sqrt[3]{81+3\sqrt{243}}$.

42. $\sqrt{R^2-2\,R^2\sqrt{2}}$.

43. $\sqrt[4]{32-64\sqrt{3}}$.

44. $\sqrt{\dfrac{R^2+R^2\sqrt{3}}{2}}$.

45. $\sqrt{R^2-\left(\dfrac{R}{2}\right)^2\sqrt{2}}$.

46. $\sqrt{R^2\sqrt{2}-\dfrac{R^2}{2}}$.

47. $\sqrt{\dfrac{R^2}{3}-R^2\sqrt{3}}$.

Reversing the process of simplification,

$$3\sqrt{2}=\sqrt{9}\sqrt{2}=\sqrt{18}.\quad \text{And } 2\sqrt[3]{3}=\sqrt[3]{8}\sqrt[3]{3}=\sqrt[3]{24}.$$

Express entirely under the radical sign:

48. $2\sqrt{2}$.

49. $3\sqrt{5}$.

50. $3\sqrt[3]{4}$.

51. $4\sqrt[3]{2}$.

52. $3\sqrt[3]{\frac{1}{3}}$.

53. $x^2\sqrt{x^3}$.

54. $2x^2\sqrt[3]{x^2}$.

55. $\frac{a}{2}\sqrt[3]{\frac{4}{a^2}}$.

56. $(a+2)\sqrt{\frac{1}{a^2-4}}$.

57. $\frac{x+3}{ax}\sqrt[3]{\frac{a^2x^2}{(x+3)^2}}$.

93. Addition and subtraction of radicals. **Similar** radicals are radicals of the same order with radicands which are identical or which can be made so by simplification.

Thus $3\sqrt[3]{ac^2}$ and $\sqrt[3]{ac^2}$ are similar radicals. Also $\sqrt{8}$ and $\sqrt{18}$ are similar, for $\sqrt{8} = 2\sqrt{2}$ and $\sqrt{18} = 3\sqrt{2}$.

Dissimilar radicals are radicals which are not similar.

The sum or difference of similar radicals can be expressed as one term, while the sum or the difference of dissimilar radicals can only be indicated.

Thus $\quad 5\sqrt{2}$ plus $3\sqrt{2} = 8\sqrt{2}$.

But $\quad 5\sqrt{2}$ plus $3\sqrt[3]{2} = 5\sqrt{2} + 3\sqrt[3]{2}$.

EXAMPLES

Simplify and collect:

1. $2\sqrt{8} + \sqrt{18} - \sqrt{50}$.

Solution: $\quad 2\sqrt{8} + \sqrt{18} - \sqrt{50}$.

Simplifying, $4\sqrt{2} + 3\sqrt{2} - 5\sqrt{2}$.

Collecting, $\quad 2\sqrt{2}$.

2. $\sqrt[3]{16a^3} + 2\sqrt[3]{54a^3} - 3\sqrt[3]{2a^3}$.

Solution: $\quad \sqrt[3]{16a^3} + 2\sqrt[3]{54a^3} - 3\sqrt[3]{2a^3}$.

Simplifying, $2a\sqrt[3]{2} + 6a\sqrt[3]{2} - 3a\sqrt[3]{2}$.

Collecting, $\quad 5a\sqrt[3]{2}$.

3. $15\sqrt{\frac{6}{5}} + \sqrt{\frac{3}{10}} - 3\sqrt{\frac{5}{6}} + \sqrt{\frac{5}{3}}$.

Solution: $\quad 15\sqrt{\frac{6}{5}} + \sqrt{\frac{3}{10}} - 3\sqrt{\frac{5}{6}} + \sqrt{\frac{5}{3}}$.

Simplifying, $3\sqrt{30} + \frac{1}{10}\sqrt{30} - \frac{1}{2}\sqrt{30} + \frac{1}{3}\sqrt{15}$.

Collecting, $\quad 2\frac{3}{5}\sqrt{30} + \frac{1}{3}\sqrt{15}$.

EXERCISES

Simplify and collect:

1. $\sqrt{27}+\sqrt{12}$.

2. $\sqrt{45}-\sqrt{20}$.

3. $2\sqrt{200}-3\sqrt{8}$.

4. $\sqrt[3]{56}+2\sqrt[3]{189}$.

5. $2\sqrt[3]{320}-\sqrt[2]{50}$.

6. $\sqrt[4]{25}-\sqrt[2]{50}$.

7. $\sqrt[4]{32}+5\sqrt[4]{162}$.

8. $5\sqrt{\frac{1}{2}}-\frac{3}{2}\sqrt{2}$.

9. $\sqrt{\frac{1}{3}}+2\sqrt{\frac{4}{3}}-3\sqrt{\frac{2}{3}}$.

10. $\sqrt{\frac{3}{10}}-\sqrt{120}-2\sqrt{\frac{6}{5}}$.

11. $\sqrt{\frac{5}{18}}+2\sqrt{\frac{32}{5}}-\sqrt{\frac{10}{9}}$.

12. $\sqrt[3]{\frac{6}{8}}-\sqrt[3]{12}+\sqrt[3]{6}$.

13. $a\sqrt[4]{4}+\sqrt{8a^2}$.

14. $R-\sqrt{\dfrac{3R^2}{4}}$.

15. $2x\sqrt[3]{54x}-3\sqrt[3]{16x^4}+\sqrt[6]{4x^2}$.

16. $\sqrt[3]{81x^7}+x\sqrt[3]{375x^4}-\sqrt[12]{16x^4}$.

17. $\sqrt{a^3bc}-a\sqrt{abc}+ac\sqrt{\dfrac{b}{ac}}$.

18. $\sqrt{(m+n)^3}-n\sqrt[4]{(m+n)^2}$.

19. $\sqrt[3]{(a+b)^4}-\sqrt[3]{8a^3(a+b)}+\sqrt[6]{a^2+2ab+b^2}$.

20. $\sqrt[2]{a^3+4a^2+4a}-\sqrt[2]{a^3}-\dfrac{2}{a^3}\sqrt{a^7}$.

21. $\sqrt{x^3y^3}-x^2y^2\sqrt{\dfrac{1}{xy}}+xy\sqrt{2+\dfrac{x^2+y^2}{xy}}$.

22. $rs\sqrt[3]{rs}+\sqrt[3]{\dfrac{1}{r^2s^2}}-2\sqrt[3]{r^4s^4}$.

23. $2\sqrt{\dfrac{y}{x}}-\sqrt{\dfrac{x}{y}}+\sqrt{2+\dfrac{x^2+y^2}{xy}}$.

24. $\sqrt{3x^2-18x+27}-\sqrt{27(x^2+2x+1)}$.

25. $\sqrt[3]{(a-3)^2(5a-15)}+\sqrt[3]{40}$.

Note. Though methods of classifying irrational expressions are found in the works of Euclid, the Hindus and the Arabs were the first to develop this part of algebra in anything like the form used to-day. The word *surd* is derived from the mistranslation of a Greek word which means, not absurd or foolish, but inexpressible, that is, inexpressible in terms of rational numbers.

94. Multiplication of radicals. Radicals of the same order are multiplied as follows:

EXAMPLES

1. Multiply $3\sqrt{8}$ by $2\sqrt{5}$.

Solution: $3\sqrt{8}\cdot 2\sqrt{5} = 6\sqrt{40} = 6\cdot 2\sqrt{10} = 12\sqrt{10}$.

2. Multiply $5\sqrt[3]{4ax^2}$ by $\sqrt[3]{2a^2x^2}$.

Solution: $5\sqrt[3]{4ax^2}\cdot\sqrt[3]{2a^2x^2} = 5\sqrt[3]{8a^3x^4} = 10ax\sqrt[3]{x}$.

3. Multiply $3\sqrt{5} - 4\sqrt{3}$ by $2\sqrt{5} + \sqrt{3}$.

Solution:

$$\begin{array}{llll}
3\sqrt{5} & -4\sqrt{3} & & \\
2\sqrt{5} & +\sqrt{3} & & \\
\hline
30 & -8\sqrt{15} & & \\
 & +3\sqrt{15} & -12 & \\
\hline
30 & -5\sqrt{15} & -12 & = 18 - 5\sqrt{15}.
\end{array}$$

4. Multiply $2\sqrt{a} + 5\sqrt{a-b}$ by $\sqrt{a} - \sqrt{a-b}$.

Solution:

$$\begin{array}{llll}
2\sqrt{a} & +5\sqrt{a-b} & & \\
\sqrt{a} & -\sqrt{a-b} & & \\
\hline
2a & +5\sqrt{a^2-ab} & & \\
 & -2\sqrt{a^2-ab} & -5(a-b) & \\
\hline
2a & +3\sqrt{a^2-ab} & -5a+5b & = 5b - 3a + 3\sqrt{a^2-ab}.
\end{array}$$

Radicals of different orders are multiplied as follows:

EXAMPLES

1. Multiply $\sqrt[2]{a}$ by $\sqrt[3]{c}$.

Solution: $\sqrt{a} = a^{\frac{1}{2}}$, and $\sqrt[3]{c} = c^{\frac{1}{3}}$.

Reducing the exponents of a and c to equivalent fractions having the least common denominator,

$$a^{\frac{1}{2}} = a^{\frac{3}{6}}, \text{ and } c^{\frac{1}{3}} = c^{\frac{2}{6}}.$$

But $$a^{\frac{3}{6}} = \sqrt[6]{a^3} \text{ and } c^{\frac{2}{6}} = \sqrt[6]{c^2}.$$

Then $$\sqrt[2]{a}\cdot\sqrt[3]{c} = \sqrt[6]{a^3}\cdot\sqrt[6]{c^2} = \sqrt[6]{a^3c^2}.$$

2. Multiply $\sqrt[3]{4}$ by $\sqrt[2]{3}$.

Solution: $\sqrt[3]{4}\cdot\sqrt[2]{3} = 4^{\frac{1}{3}}\cdot 3^{\frac{1}{2}} = 4^{\frac{2}{6}}\cdot 3^{\frac{3}{6}} = \sqrt[6]{4^2}\cdot\sqrt[6]{3^3} = \sqrt[6]{4^2\cdot 3^3} = \sqrt[6]{432}$

The method of multiplying radicals illustrated in the preceding examples may be stated in the

RULE. *If necessary, reduce the radicals to the same order.*

Find the products of the coefficients of the radicals for the coefficient of the radical part of the result.

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for the multiplication of imaginary numbers, that is, for radicals of even order in which the radicands are negative. This case will be discussed later.

EXERCISES

Perform the indicated multiplications and simplify the products:

1. $\sqrt{2} \cdot \sqrt{8}$.

2. $\sqrt{3} \cdot \sqrt{27}$.

3. $5^{\frac{1}{2}} \cdot 20^{\frac{1}{2}}$.

4. $18^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$.

5. $\sqrt[3]{16} \cdot \sqrt[3]{4}$.

6. $\sqrt[3]{4} \cdot \sqrt[3]{12}$.

7. $(100)^{\frac{1}{3}} \cdot (30)^{\frac{1}{3}}$.

8. $\sqrt[4]{8} \cdot \sqrt[4]{32}$.

9. $\sqrt{\frac{12}{25}} \cdot \sqrt{75}$.

10. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{27}{8}}$.

11. $\sqrt{11} \cdot \sqrt{\frac{1}{11}}$.

12. $a^{\frac{1}{2}} \cdot (bc)^{\frac{1}{2}}$.

13. $2\sqrt{x} \cdot \sqrt[2]{4x^3}$.

14. $5\sqrt[3]{2a} \cdot 3\sqrt[3]{16a}$.

15. $2\sqrt{Rs} \cdot 7\sqrt{r^3s^3t^2}$.

16. $\sqrt{\frac{a}{x}} \cdot \sqrt{\frac{4x}{a}}$.

17. $\sqrt{75a} \cdot (45a)^{\frac{1}{2}}$.

18. $\sqrt{2u} \cdot \sqrt{4v} \cdot \sqrt{6uv}$.

19. $5\sqrt{3m} \cdot 5\sqrt{3m}$.

20. $(3\sqrt[2]{3x})^2$.

21. $\sqrt[3]{2} \cdot \sqrt{2}$.

22. $\sqrt{3} \cdot \sqrt[3]{3}$.

23. $\sqrt[3]{2} \cdot \sqrt{3}$.

24. $\sqrt{2} \cdot \sqrt[3]{3}$.

25. $\sqrt[3]{4} \cdot \sqrt{2}$.

26. $\sqrt{8} \cdot \sqrt[3]{8}$.

27. $\sqrt{3} \cdot \sqrt[3]{24}$.

28. $\sqrt{a} \cdot \sqrt[3]{a}$.

29. $\sqrt{a^3} \cdot \sqrt[3]{a^2}$.

30. $\sqrt{2a} \cdot \sqrt[3]{2a}$.

31. $\sqrt[3]{4x^2} \cdot \sqrt{2x}$.

32. $\sqrt[3]{\frac{5x}{a}} \cdot \sqrt[2]{\frac{a^3}{5x^2}}$.

33. $(x + a\sqrt{m})\sqrt{m^3}$.

34. $(\sqrt{5a} - \sqrt{10a^2})\sqrt{5a}$.

35. $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$.

36. $(5\sqrt{3} - 4)(3\sqrt{3} + 8)$.

37. $(\sqrt{5} - \sqrt{3} - \sqrt{2})(\sqrt{5} + \sqrt{3} + \sqrt{2})$.

38. $(3\sqrt{2} + 2\sqrt{3} + \sqrt{30})(\sqrt{2} + \sqrt{3} - \sqrt{5})$.

39. $\dfrac{6a + b\sqrt{2}}{2} \cdot \dfrac{6a + b\sqrt{2}}{2}$.

40. $\left(R - \dfrac{R}{2}\sqrt{3}\right)\left(2R + \dfrac{3R}{2}\sqrt{3}\right)$.

41. $\sqrt{a+b} \cdot \sqrt{a-b} \cdot \sqrt{2a^2 - 2b^2}$.

42. $(2\sqrt{x-a} + \sqrt{a})(-3\sqrt{x-a} - 5\sqrt{a})$.

43. $(3\sqrt[3]{x-a})(-5\sqrt[3]{x-a})$.

44. $\sqrt{2 - \sqrt{2}} \cdot \sqrt{2 + \sqrt{2}}$.

45. $\sqrt{a - \sqrt{b}} \cdot \sqrt{2a - 2\sqrt{b}}$.

46. $\sqrt{\dfrac{R}{2}\sqrt{5} + R} \cdot \sqrt{\dfrac{R}{2}\sqrt{5} - R}$.

Square :

47. $\sqrt[3]{2}$.
48. $2\sqrt[3]{3}$.
49. $2\sqrt[3]{12}$.
50. $\sqrt{2 - \sqrt{2}}$.
51. $3\sqrt{x + \sqrt{3}}$.
52. $(2\sqrt[3]{3})^2$.
53. $2a\sqrt[3]{8x}$.
54. $\sqrt[3]{2} + \sqrt[3]{3}$.
55. $\sqrt[3]{4 + 4\sqrt{3}}$.
56. $\frac{1}{3}\sqrt[3]{9 - 9\sqrt{2}}$.
57. $\sqrt{2} + \sqrt[3]{3}$.
58. $2\sqrt[4]{3 - \sqrt{2}}$.

Cube :

59. $2\sqrt[2]{3}$.
60. $3\sqrt[2]{2}$.
61. $(\sqrt{2})^2$.
62. $(\sqrt[2]{3})^3$.
63. $\sqrt{3} - \sqrt{6}$.
64. $2\sqrt{2} + \sqrt{3}$.
65. $\sqrt[3]{3} - \sqrt[3]{2}$.
66. $\sqrt{2 + \sqrt{2}}$.
67. $\sqrt[3]{2} - \sqrt{3}$.

Simplify :

68. $R^2 + \left(\dfrac{R}{2}\sqrt{2 - \sqrt{2}}\right)^2$.

69. $\sqrt{R^2 - \left(\dfrac{R}{2}\sqrt{5} - \dfrac{R}{2}\right)^2}$.

70. $\left[R^2 - \left(\dfrac{R\sqrt{5} - R}{4}\right)^2\right]^{\frac{1}{2}}$.

71. $\left[\left(R - \dfrac{R}{2}\sqrt{3}\right)^2 + \left(\dfrac{R}{2}\right)^2\right]^{\frac{1}{2}}$.

72. $\left[R^2 - \left(\dfrac{R\sqrt{2 - \sqrt{3}}}{2}\right)^2\right]^{\frac{1}{2}}$.

73. $\sqrt{R^2 - \left(\dfrac{R\sqrt{2 - \sqrt{2}}}{2}\right)^2}$.

74. $\sqrt{\left(\frac{R\sqrt{2}}{2}\right)^2+\left(R-\frac{R}{2}\sqrt{2}\right)^2}.$

75. $\left(\frac{R}{2}\sqrt{2+\sqrt{2}}\right)\left(\frac{R\sqrt{2-\sqrt{2}}}{2}\right)4.$

76. Find the value of x^2+4x+1 if $x=-2+\sqrt{3}$.

77. Find the value of x^2-2x-3 if $x=2+\sqrt{5}$.

78. Find the values of x^2-4x-1 if $x=2\pm\sqrt{5}$.

79. Find the values of $3x^2+3x-5$ if $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{13}$.

80. Do the values $x=2\pm\sqrt{3}$ satisfy the equation $x^2-4x+1=0$?

81. Do the values $x=-4\pm\sqrt{5}$ satisfy the equation $x^2+8x+11=0$?

82. What inference seems warranted as a result of Exercise 80? of Exercise 81?

95. Division of radicals. It is frequently necessary to find the approximate value of an expression which involves division by a radical expression. Thus $2\div\sqrt{3}$, $(4-\sqrt{3})\div(2-\sqrt{3})$, $\frac{\sqrt{3}}{\sqrt{5}}$, and $\frac{3\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ are types which often occur.

To find the approximate value of $2\div\sqrt{3}$, we may extract the square root of 3 to several decimal places and then divide 2 by the approximate root obtained. Both of these processes are long and one of them is unnecessary.

For, writing $2\div\sqrt{3}$ in the form $\frac{2}{\sqrt{3}}$ and multiplying both terms of the fraction by $\sqrt{3}$ gives $\frac{2\sqrt{3}}{3}$. The process of finding the approximate value of $\frac{2\sqrt{3}}{3}$ involves but one long operation.

Similarly the process of finding the approximate value of $\sqrt{7}\div(\sqrt{7}-\sqrt{2})$ involves three rather lengthy operations, — the extracting of two square roots, and one long division. The labor of two of these operations can be avoided.

Evidently $\sqrt{7} \div (\sqrt{7} - \sqrt{2}) = \dfrac{\sqrt{7}}{\sqrt{7} - \sqrt{2}}$. Multiplying both terms of this fraction by $\sqrt{7} + \sqrt{2}$ gives $\dfrac{\sqrt{7}(\sqrt{7} + \sqrt{2})}{(\sqrt{7} - \sqrt{2})(\sqrt{7} + \sqrt{2})}$, or $\dfrac{7 + \sqrt{14}}{7 - 2}$, or $\dfrac{7 + \sqrt{14}}{5}$. Finding the value of $\dfrac{7 + \sqrt{14}}{5}$ involves only one long operation, extracting the square root of 14.

As in the two preceding illustrations, division of radicals is usually an indirect process performed by means of a **rationalizing** factor of the divisor.

One radical expression is a **rationalizing factor** for another if the *product* of the two is *rational.*

A rationalizing factor for $\sqrt{3}$ is $\sqrt{3}$, for $\sqrt{3} \cdot \sqrt{3} = 3$.

For $\sqrt[3]{2}$ a rationalizing factor is $\sqrt[3]{4}$, since $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$.

Similarly $\sqrt{7} - \sqrt{2}$ is a rationalizing factor for $\sqrt{7} + \sqrt{2}$, as their product $(\sqrt{7} - \sqrt{2})(\sqrt{7} + \sqrt{2}) = 7 - 2 = 5$.

In like manner $(3\sqrt{5} - 2\sqrt{3})(3\sqrt{5} + 2\sqrt{3}) = 45 - 12 = 33$. Therefore $3\sqrt{5} - 2\sqrt{3}$ is a rationalizing for $3\sqrt{5} + 2\sqrt{3}$.

The binomial radicals of the last two illustrations are of the general types $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$. Such binomials are called **conjugate** radicals and *either* is a rationalizing factor for the other. If a and b are rational, the product $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$, or $a - b$, is a *rational* number.

There are many other types of radical expressions which have rationalizing factors. They seldom arise, however, and are too difficult for treatment in elementary algebra.*

An irrational expression may have more than one rationalizing factor. Thus $\sqrt{18}$, $\sqrt{8}$, and $\sqrt{2}$ are rationalizing factors for $\sqrt{18}$. For $\sqrt{18} \cdot \sqrt{18} = 18$; and $\sqrt{18} \cdot \sqrt{8} = \sqrt{144} = 12$; and $\sqrt{18} \cdot \sqrt{2} = \sqrt{36} = 6$. Similarly, since $\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$, and $\sqrt[3]{4} \cdot \sqrt[3]{16} = \sqrt[3]{64} = 4$, both $\sqrt[3]{2}$ and $\sqrt[3]{16}$ are rationalizing factors for $\sqrt[3]{4}$. In practice it is best to choose for monomials the rationalizing factor which has the least radicand.

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* See Hawkes's "Advanced Algebra," page 62.

EXERCISES

Determine a rationalizing factor for each of the following expressions and find the product of the expression and the factor:

1. $\sqrt{5}$.
2. $3\sqrt{6}$.
3. $2\sqrt{7}$.
4. $\sqrt{8}$.
5. $\sqrt{32}$.
6. $\sqrt{27}$.
7. $\sqrt[3]{2}$.
8. $\sqrt[3]{3}$.
9. $2\sqrt[3]{5}$.
10. $\sqrt[3]{16}$.
11. $\sqrt[3]{25}$.
12. $\sqrt[3]{36}$.
13. $\sqrt[3]{49}$.
14. $\sqrt{2}+3$.
15. $\sqrt{3}-\sqrt{2}$.
16. $3+\sqrt{7}$.
17. $3\sqrt{2}-5$.
18. $4\sqrt{3}-\sqrt{2}$.
19. $2\sqrt{5}+7\sqrt{6}$.
20. $\sqrt{x}-\sqrt{a}$.
21. $\sqrt{3a}+\sqrt{x}$.
22. $3\sqrt{x}-a\sqrt{2}$.
23. $\sqrt{x+a}+\sqrt{x}$.
24. $\sqrt{x}-\sqrt{a-x}$.
25. $\sqrt{2ax-x^2}-\sqrt{ax}$.

The usefulness of rationalizing factors is illustrated in Examples 4–8 which follow.

The student should now study Examples 1–6, pages 251–252, and the rule on pages 252–253, and then solve Exercises 1–32, pages 253–254.

EXAMPLES

1. Divide $\sqrt{6}$ by $\sqrt{2}$.

Solution: By direct division, $\sqrt{6}\div\sqrt{2}=\dfrac{\sqrt{6}}{\sqrt{2}}=\sqrt{\dfrac{6}{2}}=\sqrt{3}$.

2. Divide $6\sqrt{5}$ by $3\sqrt{3}$.

Solution: By direct division, coefficient by coefficient and radicand by radicand, $6\sqrt{5}\div 3\sqrt{3}=\dfrac{6\sqrt{5}}{3\sqrt{3}}=2\sqrt{\dfrac{5}{3}}$, which becomes $\dfrac{2}{3}\sqrt{15}$.

3. Divide $8\sqrt{12}$ by $2\sqrt[3]{6}$.

Solution: Reducing the surds of the same order, and then proceeding as in direct division,

$$8\sqrt{12}\div 2\sqrt[3]{6}=\frac{8(12)^{\frac{1}{2}}}{2(6)^{\frac{1}{3}}}=\frac{4(12)^{\frac{3}{6}}}{(6)^{\frac{2}{6}}}=4\sqrt[6]{\frac{12\cdot 12\cdot 12}{6\cdot 6}}$$
$$=4\sqrt[6]{2\cdot 2\cdot 12}=4\sqrt[6]{48}.$$

If the monomial divisor is a surd, it is always possible and often far more convenient to divide by means of a rationalizing factor of the divisor.

4. Divide 4 by $\sqrt{3}$.

Solution: $4 \div \sqrt{3} = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{\sqrt{3}\sqrt{3}} = \dfrac{4\sqrt{3}}{3}.$

5. Divide 6 by $\sqrt[3]{3}$.

Solution: $6 \div \sqrt[3]{3} = \dfrac{6}{\sqrt[3]{3}} = \dfrac{6\sqrt[3]{9}}{\sqrt[3]{3}\cdot\sqrt[3]{9}} = \dfrac{6\sqrt[3]{9}}{\sqrt[3]{27}} = \dfrac{6\sqrt[3]{9}}{3} = 2\sqrt[3]{9}.$

6. Divide $\sqrt{3}$ by $\sqrt[3]{2}$.

Solution: $\sqrt{3} \div \sqrt[3]{2} = \dfrac{\sqrt{3}}{\sqrt[3]{2}} = \dfrac{\sqrt{3}\cdot\sqrt[3]{4}}{\sqrt[3]{2}\cdot\sqrt[3]{4}} = \dfrac{3^{\frac{1}{2}}\cdot 4^{\frac{1}{3}}}{2} = \dfrac{3^{\frac{3}{6}}\cdot 4^{\frac{2}{6}}}{2}$

$$= \tfrac{1}{2}\sqrt[6]{3^3\cdot 4^2} = \tfrac{1}{2}\sqrt[6]{432}.$$

When the divisor is a binomial (or polynomial) radical, the practical method of division is an indirect method by means of a rationalizing factor.

7. Divide 8 by $3 + \sqrt{7}$.

Solution: $8 \div (3 + \sqrt{7}) = \dfrac{8}{3+\sqrt{7}} = \dfrac{8(3-\sqrt{7})}{(3+\sqrt{7})(3-\sqrt{7})}$

$$= \frac{24 - 8\sqrt{7}}{9-7} = 12 - 4\sqrt{7}.$$

8. Divide $\sqrt{5} + \sqrt{3}$ by $2\sqrt{5} - \sqrt{3}$.

Solution: $(\sqrt{5}+\sqrt{3}) \div (2\sqrt{5}-\sqrt{3})$

$$= \frac{\sqrt{5}+\sqrt{3}}{2\sqrt{5}-\sqrt{3}} = \frac{(\sqrt{5}+\sqrt{3})(2\sqrt{5}+\sqrt{3})}{(2\sqrt{5}-\sqrt{3})(2\sqrt{5}+\sqrt{3})}$$

$$= \frac{10 + 3\sqrt{15} + 3}{20-3} = \frac{13}{17} + \frac{3}{17}\sqrt{15}.$$

For division of radicals we may use the

RULE. *Write the dividend over the divisor in the form of a fraction.*

Then multiply the numerator and denominator of the fraction by the rationalizing factor of the denominator and simplify the resulting fraction.

Every irrational algebraic expression containing nothing more complicated than rational numbers and radicals has a rationalizing factor. To find this factor for any given irrational expression is a problem which requires considerable algebraic training. At the present time it is wholly beyond the student to find the rationalizing factor of even so simple an expression as the denominator of the fraction $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt[3]{2}+\sqrt[4]{2}}$. The approximate value of such a fraction can be obtained, however, by dividing the sum of the approximate values of the roots in the numerator by the sum of the approximate values of the roots in the denominator.

EXERCISES

Perform the indicated division:

1. $\sqrt{10}\div\sqrt{2}$.

2. $(18)^{\frac{1}{2}}\div(3)^{\frac{1}{2}}$.

3. $\frac{\sqrt{6}}{\sqrt{18}}$.

4. $6\div 2\sqrt{2}$.

5. $\frac{3\sqrt{12}}{\sqrt{6}}$.

6. $8\sqrt{15}\div 4\sqrt{5}$.

7. $\frac{10}{2\sqrt{5}}$.

8. $\frac{4\sqrt{10}}{8\sqrt{5}}$.

9. $8\div 4\sqrt{3}$.

10. $\frac{3\sqrt{2}}{15\sqrt{8}}$.

11. $(\sqrt{6}+\sqrt{18})\div 3\sqrt{2}$.

12. $(\sqrt{12}-\sqrt{24})\div 2\sqrt{3}$.

13. $\frac{6\sqrt{10}+4\sqrt{15}-\sqrt{20}}{2\sqrt{5}}$.

14. $(12+\sqrt{3}+\sqrt{5})\div\sqrt{6}$.

15. $\frac{\sqrt{6}-\sqrt{9}+18}{2\sqrt{2}}$.

16. $(\sqrt{5}+2)\div\sqrt{125}$.

17. $\sqrt{\frac{32}{10}}\div\sqrt{\frac{4}{5}}$.

18. $(xy)^{\frac{1}{2}}\div x^{\frac{1}{2}}$.

19. $a\sqrt{bc}\div d\sqrt{c}$.

20. $\frac{a^2\sqrt{c}}{a\sqrt{bc}}$.

21. $\sqrt[3]{16}\div\sqrt[3]{4}$.

22. $8\sqrt[3]{125}\div 4\sqrt[3]{25}$.

23. $\frac{2\sqrt[3]{5}}{3\sqrt[3]{4}}$.

24. $a\,9^{\frac{1}{3}}\div b\,8^{\frac{1}{3}}$.

25. $\sqrt[3]{\frac{1}{8}}\div\sqrt[3]{\frac{1}{4}}$.

26. $\sqrt[3]{4}\div\sqrt[2]{2}$.

27. $\sqrt{2}\div\sqrt[3]{4}$.

28. $4^{\frac{1}{3}} \div 6^{\frac{1}{2}}$.

29. $6^{\frac{1}{2}} \div 4^{\frac{1}{3}}$.

30. $\sqrt[5]{27} \div \sqrt{27}$.

31. $\sqrt[4]{\frac{1}{8}} \div \sqrt[3]{\frac{1}{2}}$.

32. $\sqrt[6]{\dfrac{a^2}{72\,a}} \div \sqrt[3]{\dfrac{a}{81}}$.

33. $5 \div (\sqrt{5} + 2)$.

Hint. Study Examples 7 and 8, page 252.

34. $\dfrac{2}{2 - \sqrt{3}}$.

35. $4 \div (\sqrt{3} - \sqrt{2})$.

36. $\sqrt{5} \div (3\sqrt{2} - \sqrt{5})$.

37. $(\sqrt{5} - \sqrt{3}) \div (\sqrt{5} + \sqrt{3})$.

38. $\dfrac{4\sqrt{3} + 2\sqrt{2}}{4\sqrt{3} - 2\sqrt{2}}$.

39. $\dfrac{(2\sqrt{5} - 3\sqrt{3})}{5\sqrt{5} - 5}$.

Find to three decimals the approximate values of the following:

40. $3 + \sqrt{2}$.

41. $14 - 5\sqrt{7}$.

42. $6 \pm 2\sqrt{5}$.

43. $\dfrac{7 \pm \sqrt{6}}{3}$.

44. $3\sqrt{6} \div 2\sqrt{5}$.

45. $\dfrac{2\sqrt{5} + 1}{3\sqrt{5} - \sqrt{3}}$.

46. $\sqrt{2 - \sqrt{3}}$.

47. $\dfrac{\sqrt{3}}{\sqrt{3} + \sqrt[3]{4} + \sqrt[4]{5}}$, given $\sqrt[3]{4} = 1.5874$.

Change the following fractions to equivalent fractions having rational denominators:

48. $\dfrac{(\sqrt{a} + \sqrt{b})}{(\sqrt{a} - \sqrt{b})}$.

49. $\dfrac{2\sqrt{x} - \sqrt{a}}{\sqrt{x} + 3\sqrt{a}}$.

50. $\dfrac{(r\sqrt{3} + \sqrt{r})}{\sqrt{3} - \sqrt{r}}$.

51. $\dfrac{m\sqrt{n} + a\sqrt{b}}{m\sqrt{n} - a\sqrt{b}}$.

52. $\dfrac{\sqrt{a - 2} - 2}{\sqrt{a - 2} + 2}$.

53. $\dfrac{2}{\sqrt[4]{2 + \sqrt{2}}}$.

Perform the indicated division:

54. $(x - \sqrt{a + b}) \div (x + \sqrt{a + b})$.

55. $(\sqrt{a} + \sqrt{b}) \div (\sqrt{c} + \sqrt{d})$.

56. Is there any real distinction between the direction which precedes Exercise 48 and that which precedes Exercise 54?

PROBLEMS

(Obtain answers in *simplest radical form.*)

1. One leg of a right triangle is 10 and the other is 5. Find the hypotenuse.

2. The hypotenuse of a right triangle is 10 and one leg is 5. Find the other leg and the area.

3. The hypotenuse of a right triangle is R and one leg is $\frac{R}{2}$. Find the other leg and the area.

4. Find the diagonal of a square whose side is 10.

5. Find the sides and the area of a square whose diagonal is 10.

6. Find the sides and the area of a square whose diagonal is $2R$.

7. The side of an equilateral triangle is 12. Find the altitude and the area.

8. The side of an equilateral triangle is S. Find the altitude and the area.

9. The altitude of an equilateral triangle is 10. Find the side and the area.

10. The legs of a right triangle are equal. Its hypotenuse is 20. Find the legs and the area of the triangle.

11. The legs of a right triangle are equal and its area is 32. Find the hypotenuse.

12. The legs of a right triangle are $\frac{R}{2}$ and $\frac{3R}{2}$. Find the hypotenuse.

13. One leg of a right triangle is $\frac{R}{5}$. The hypotenuse is R. Find the other leg.

14. The legs of a right triangle are R and $\frac{R}{2}(\sqrt{5}-1)$. Find the hypotenuse.

15. The legs of a right triangle are $\frac{R}{2}$ and $R-\frac{R}{2}\sqrt{3}$. Find the hypotenuse.

16. The base of a certain rectangle is $\frac{2R}{3}\sqrt{4-\sqrt{3}}$ and the altitude is $9R\frac{\sqrt{4+\sqrt{3}}}{10}$. Find the area of a second rectangle five times as long and three times as wide as the first.

96. Factors involving radicals. In the chapter on factoring it was definitely stated that factors involving radicals would not then be considered. This limitation on the character of a factor is no longer necessary. Consequently many expressions which previously have been regarded as prime may now be thought of as factorable. Thus $3x^2-1=(\sqrt{3}x+1)(\sqrt{3}x-1)$ and $4x^2-5=(2x+\sqrt{5})(2x-\sqrt{5})$.

It is not usual to allow the variable in an expression to occur under a radical sign in the factors. Hence, if x is a variable, the trinomial x^2+x+1 is not regarded as factorable into $(x+\sqrt{x}+1)(x-\sqrt{x}+1)$, though the student can easily show that $(x+\sqrt{x}+1)(x-\sqrt{x}+1)=x^2+x+1$.

Therefore in this extension of our notion of a factor it must be clearly understood that the use of radicals is limited to the coefficients in the terms of the factors. Such a conception of a factor is a necessity for certain work in advanced algebra and geometry, and is very desirable in solving equations by factoring.

To restrict the use of radicals in the way just indicated is necessary for the sake of definiteness. Otherwise it would be impossible to obey a direction to factor even so simple an expression as x^2-y^2; for if the variable is allowed under a radical sign in a factor, x^2-y^2 has an infinite number of factors.

Thus
$$\begin{aligned} x^2-y^2 &= (x+y)(x-y) \\ &= (x+y)(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) \\ &= (x+y)(\sqrt{x}+\sqrt{y})(\sqrt[4]{x}+\sqrt[4]{y})(\sqrt[4]{x}-\sqrt[4]{y}) \\ &= \text{etc.} \end{aligned}$$

The extension of factoring explained above can be applied to the solution of equations as follows:

EXAMPLE

Solve $x^2 - 7 = 0$ by factoring.

Solution: $$x^2 - 7 = 0. \tag{1}$$

Then $$(x + \sqrt{7})(x - \sqrt{7}) = 0. \tag{2}$$

Therefore $$x + \sqrt{7} = 0, \text{ or } x = -\sqrt{7},$$

and $$x - \sqrt{7} = 0, \text{ or } x = \sqrt{7}.$$

It is apparent at once that these values check in $x^2 - 7 = 0$.

In (2) it is obvious that if a root be substituted for the variable, one of the factors must become zero.

EXERCISES

Factor:

1. $x^2 - 6$.

2. $3x^2 - 4$.

3. $5x^2 - 1$.

4. $x^4 - 4$.

5. $4x^4 - 1$.

6. $x^3 - 2$.

7. $x^3 + 6$.

8. $3x^3 - 1$.

9. $x^3 + 4$.

10. $2x^3 + 8$.

11. $2x^3 - 8$.

12. $6x^3 + 24$.

Solve by factoring and check:

13. $x^2 - 2 = 0$.

14. $x^2 - 6 = 0$.

15. $2x^2 - 1 = 0$.

16. $x^4 + 6 = 5x^2$.

17. $4x^4 + 5 = 12x^2$.

18. $3x^4 + 8 = 14x^2$.

19. $5x^4 - 16x^2 + 3 = 0$.

20. $x^4 + 8a = 4x^2 + 2ax^2$.

21. $ax^4 - x^2 + 3a = 3a^2x^2$.

22. $4x^4 + a = x^2 + 4ax^2$.

Biographical Note. FRANÇOIS VIETA. The reason that algebra is a universal language which does not depend entirely on the nationality of the writer lies in the fact that the symbols used to indicate the various operations and relations are widely understood and adopted. This has not always been the case, and for a long time during the early history of the subject there was no accepted notation in algebra, but each man used any symbol that suited him. One of the men who did most to establish a fixed notation was François Vieta (1540–1603), a French lawyer who studied and wrote on mathematics as a pastime. He was in public life during his whole career, and was well known for his ability to decipher the hidden meaning of dispatches captured from the enemy.

It was he who established the use of the signs + and − for addition and subtraction, which, to be sure, had been used before his time, but were not generally accepted. He also denoted the known numbers in an equation by the consonants, B, C, D, etc., and the unknowns by the vowels A, E, I, etc. He also recognized the existence of negative roots of equations, but rejected them as absurd.

To denote the second and third powers of the unknown, he used the letters Q (*quadratus*) and C (*cubus*) respectively. Instead of using the sign $=$, he wrote aeq. (*aequalis* or *aequatur*). Thus Vieta would have written the equation $x^3 - 8x^2 + 16 = 40$ in the form

$$1\,C - 8\,Q + 16\,N \text{ aeq. } 40.$$

Before the time of Vieta this equation would have been written in a much more primitive notation. For instance, with writers only a little earlier it would appear as

$$\text{Cubus } \bar{m} \text{ 8Census } \bar{p} \text{ 16 rebus aequatur 40.}$$

It is easily seen that operations on equations in this form would be very hard to perform.

Vieta is further distinguished as being the first man to obtain an exact numerical expression for the number π, which occurs in geometry. His form of expression calls for an infinite number of operations which, of course, could never be performed, but the further one proceeded, the closer would be the approximation obtained. In a certain sense the familiar sign $\sqrt{\ }$ implies an infinite number of operations, for one can never go through the process of extracting the square root of 2, for instance, and come out even. Vieta's method of denoting π was, however, more involved than this, and made use of complicated irrational fractions.

FRANÇOIS VIETA

CHAPTER XXII

GRAPHICAL SOLUTION OF EQUATIONS IN ONE UNKNOWN

97. Graph of a linear function. An algebraic expression involving one or more letters is a **function** of the letter or letters involved.

Thus $2x + 3$ and $x^2 + 5x - 6$ are functions of one letter, x; $x^2 - 2xy + y^2$ and $x^3 + y^3$ are functions of two letters, x and y.

The letters of a function are usually referred to as **variables.**

A function is called **linear, quadratic,** or **cubic** according as its degree with respect to the variable (or variables) is first, second, or third respectively.

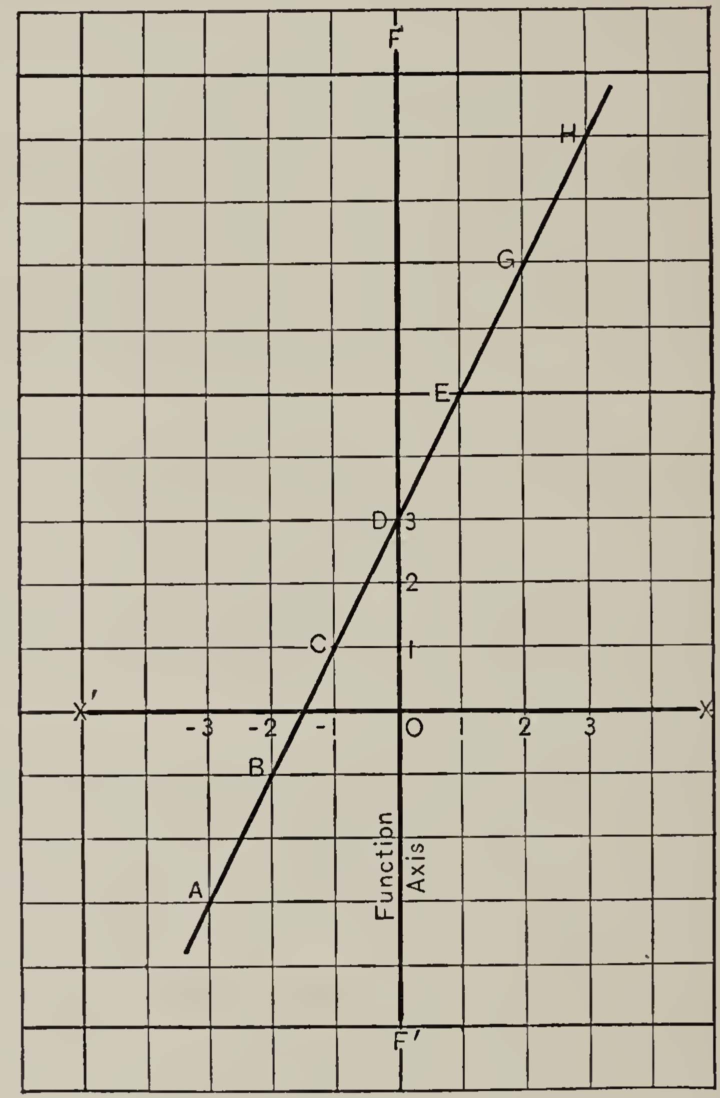
After a function of any variable, say x, has once been given, it is convenient and usual to refer to it later in the same discussion by the symbol $f(x)$, which is read *the function of* x, or, more briefly, *f of x*.

The numerical value of a linear function of x changes with every change in the variable.

Thus if $x = 1$, the linear function of x, $2x + 3$, equals $2 + 3$, or 5; if $x = 2$, $2x + 3$ equals $4 + 3$, or 7. The following table illustrates this change further.

When $x =$	-3	-2	-1	0	1	2	3
$f(x)$, $2x + 3 =$	-3	-1	1	3	5	7	9

The relation between x and the function $2x + 3$ may be represented graphically if OX continue to be the x-axis and OF (or the function axis) replace the y-axis of our previous graphical work. Beginning with $(-3, -3)$ and plotting the points corresponding to the numbers in the table, we locate points A, B, C, D, E, G, and H respectively. The *graph* of the function $f(x) = 2x + 3$ is evidently the *straight line AH.*



Graph of $f(x) = 2x + 3$

This process of plotting the relation between a function and the variable contained in the function is called *graphing the function.*

Care should always be taken in graphing to join the plotted points by a *smooth* curve or by a *straight* line, as the case may be. If the graph is not regular and graceful, it is almost certain that an error has been made in plotting the points. No equations in this book have graphs that are part straight line and part curve, or that present erratic changes in curvature. Although such curves have equations, they are usually very complicated.

EXERCISES

(Exercises 1–8 refer to the preceding graph.)

1. Read from the graph the value of x when $f(x)$ is zero.

2. Set $2x + 3$ equal to zero, and solve.

3. Compare the results of Exercises 1 and 2.

4. Read from the graph the value of x when $f(x)$ is 4.

5. Set $2x + 3$ equal to 4, and solve.

6. Compare the results of Exercises 4 and 5.

7. Can the value of x in $2x + 3 = 6$ be read from the graph? If so, read it.

8. Read from the graph the root of $2x + 3 = -2$.

9. Graph the function $f(x) = 5 - 2x$.

10. Can the root of $5 - 2x = 0$ be read from the graph just obtained? If so, read it.

11. From the graph of Exercise 9 read the roots of:

(*a*) $5 - 2x = 9$; (*c*) $5 - 2x = -3$;
(*b*) $5 - 2x = 5$; (*d*) $5 - 2x = -7$.

12. Check by substitution the roots obtained in Exercise 11.

13. What kind of a line do you expect the graph of any *linear function* of x to give?

98. Graph of a quadratic function. The quadratic function $f(x) = 4x^2 - 4x - 15$ may be graphed as follows:

When $x = 1$, $4x^2 - 4x - 15 = 4 - 4 - 15 = -15$, or $f(x) = -15$. In like manner, the other numbers in the following table can be obtained.

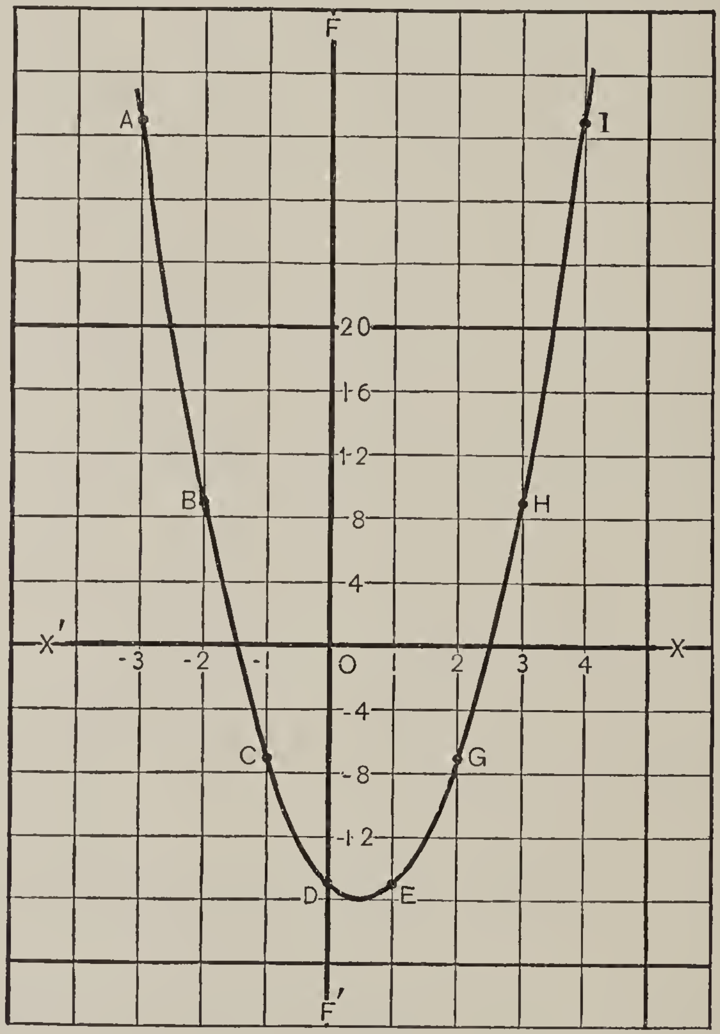
When $x =$	-3	-2	-1	0	1	2	3	4
$f(x)$, $4x^2 - 4x - 15 =$	33	9	-7	-15	-15	-7	9	33

To represent the numbers in the preceding table conveniently, it is necessary to use *different scales* for x and $f(x)$.

The difference can be seen from the numbers along the axes in the following figure.

If we begin with (– 3, 33) and plot the points corresponding to the numbers in the table, we get the points *A*, *B*, *C*, *D*, *E*, *G*, *H*, and *I* respectively.

Drawing a smooth curve through these points gives the graph of the following figure. This curve is called a **parabola**.



Graph of $f(x) = 4x^2 - 4x - 15$

The student should note that it is often best to represent values of $f(x)$ on a different scale from values of x. He should always inspect his table of values and decide what scale to use before plotting a single point. The scale should be as large as possible and yet show enough of the curve to indicate its shape clearly.

It will often be found convenient not to put the intersection of the axes in the center of the page.

EXERCISES

(Exercises 1, 4, 7, and 9 refer to the preceding graph.)

1. Read from the graph the values of x for which $4x^2 - 4x - 15$ equals zero.

2. Set $4x^2 - 4x - 15$ equal to zero, and solve by factoring.

3. Compare the answers to Exercises 1 and 2.

4. Read from the graph the value of x for which $f(x)$ equals 20.

5. What, then, are the roots of $4x^2 - 4x - 15 = 20$?

6. Solve $4x^2 - 4x - 15 = 20$ by factoring, and check your answers to Exercise 5.

7. Read from the graph the roots of:

(*a*) $4x^2 - 4x - 15 = 9$. (*b*) $4x^2 - 4x - 15 = -7$.

8. Check your answers to Exercise 7 by solving the equations (*a*) and (*b*) by factoring.

9. Can you read from the graph the value of x which makes $f(x)$ equal -25? equal -20? Explain.

10. Graph the function $x^2 - 2x - 4$.

First fill out the table:

When $x =$	-4	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$, $x^2 - 2x - 4 =$				-1	-4	-5					

Then plot the eleven points and draw through them as smooth a curve as possible.

11. Is the curve obtained in Exercise 10 similar in shape to that of the preceding figure?

12. Read from the graph of Exercise 10 the approximate values of x which make the function $x^2 - 2x - 4$ equal zero.

13. What are the roots of the equation $x^2 - 2x - 4 = 0$?

14. Can you solve the equation $x^2 - 2x - 4 = 0$ by factoring? graphically?

15. If the terms of a quadratic equation be transposed so that the second member is zero, and then the function in the first member be graphed, can the roots of the original equation be read from the curve thus obtained? Explain.

Solve Exercises 16 and 17 graphically and check each by substituting in the original equation; or solve the equation by factoring, and compare results with those obtained graphically.

16. $x^2 - 3x = 4.$ **17.** $x^2 - 4x + 4 = 0.$

18. What peculiarity has the curve obtained in Exercise 17? How many roots has $x^2 - 4x + 4 = 0$? What values of x make $x^2 - 4x + 4$ equal zero? equal to $+4$? to -1? to -10?

Exercises 19 and 20 cannot be solved by the factoring previously explained. They have irrational roots, but the approximate values of the roots can be obtained graphically. Solve Exercises 19 and 20 graphically.

19. $x^2 - 2x = 2.$ **20.** $x^2 + 2x = 5.$

21. Graph the linear function $3x + 4$, using the same scale for x and $f(x)$; then graph the function, using different scales. Compare the two lines obtained and the values of x and $f(x)$ where each graph crosses the x-axis and the F-axis respectively.

22. Proceed as in Exercise 21 with the quadratic function $x^2 + 3x + 1$.

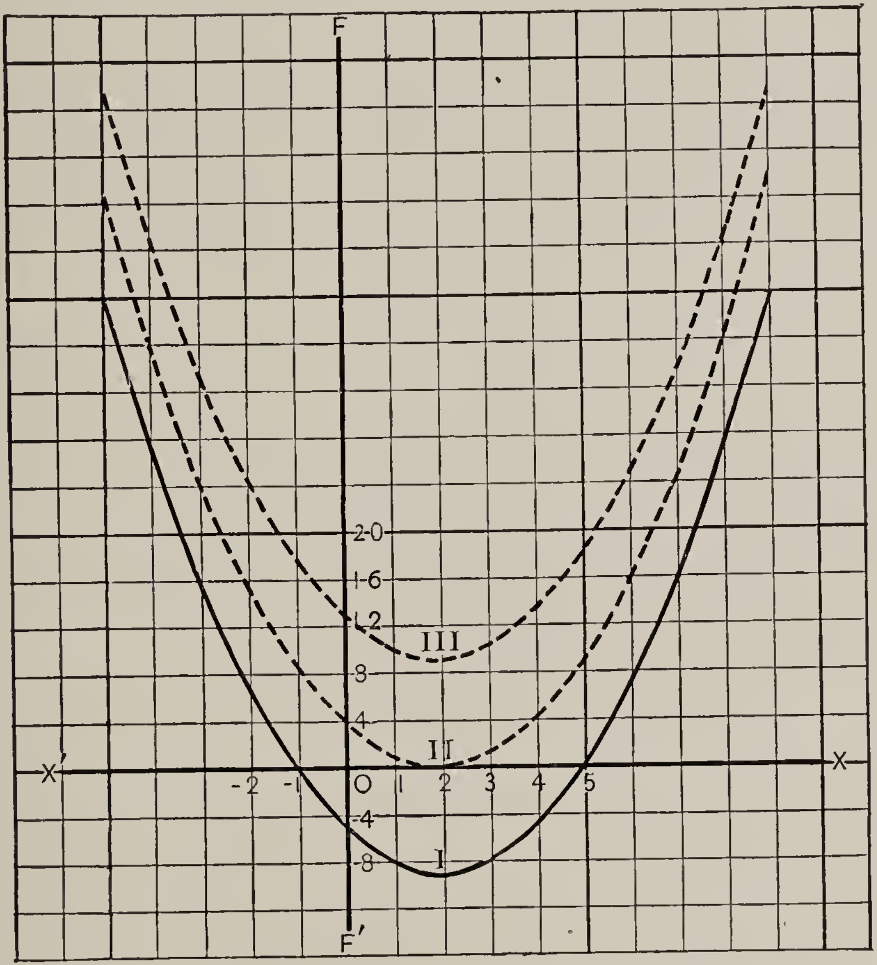
99. Graphical illustration of imaginary roots. The solution of $x^2 + 1 = 0$ gives $x = \pm\sqrt{-1}$. These roots are *imaginary* numbers. Just now it is desirable to know that many quadratic equations have imaginary roots. It will be instructive also to see the result of an attempt at a graphical solution of a quadratic equation whose roots are imaginary. For this purpose we shall consider the three equations:

$$\text{I. } x^2 - 4x - 5 = 0.$$

$$\text{II. } x^2 - 4x + 4 = 0.$$

$$\text{III. } x^2 - 4x + 13 = 0.$$

The graphs of the functions in the left members of I, II, and III are given in the adjacent figure. The three functions differ only in their constant terms; for 9 added to the constant term of I gives the constant term of II, and 9 added to the constant term of II gives the constant term of III. Apparently, as the constant term is increased, the graph rises. It does not change its shape, nor does it move to the left nor to the right.



From the graph the roots of $x^2 - 4x - 5 = 0$ are seen to be 5 and -1. These results are easily obtained by factoring: $x^2 - 4x - 5 = 0$, or $(x - 5)(x + 1) = 0$. Therefore $x = 5$ or -1.

If we imagine curve I to move upward, the two roots change in value and become the single root of curve II, which touches the x-axis at a point where x equals 2. Solving $x^2 - 4x + 4 = 0$ by factoring gives $(x - 2)(x - 2) = 0$. Therefore $x = 2$.

If we now imagine curve I to move still farther upward from its position II, it will no longer cut the x-axis. Further, we do not expect that the graph when it reaches the position of III will show the roots of the equation $x^2 - 4x + 13 = 0$, as in fact it does not. The graph does show, however, that the value of $x^2 - 4x + 13$ at the lowest point of the curve is 9. This means that for *every real value* of x, positive or negative, $x^2 - 4x + 13$ is *never less than* 9. The graph of III, then, makes clear that no real number, if substituted for x, will make $x^2 - 4x + 13$ equal to zero.

In the next chapter an algebraic method of solving any quadratic is given. That method will later be used to show the roots of $x^2 - 4x + 13 = 0$ to be the imaginary expressions $2 + 3\sqrt{-1}$ and $2 - 3\sqrt{-1}$.

The preceding explanations show that the *real* roots of a quadratic equation can be obtained by the following steps.

1. Transpose the terms so that the second member is zero.
2. Graph the function in the first member.
3. Note the x-distance of each point at which the curve crosses (or touches) the x-axis. *The values of these x-distances are the roots of the quadratic.*

If the graph obtained in 2 does not cross the x-axis, the student knows that the roots are not real. He may still regard such equations as having roots and refer to them as *imaginary*. Later he will learn precisely what an imaginary number is, and how to solve any quadratic whose roots are imaginary.

EXERCISES

If possible, solve graphically:

1. $x^2 - 6x + 7 = 0$.

2. $x^2 - 6x + 11 = 0$.

3. $2x^2 + x + 1 = 0$.

4. $2x^2 + x - 1 = 0$.

CHAPTER XXIII

QUADRATIC EQUATIONS

100. Solution by completing the square. Before taking up the work which follows, the student should review the exercises in forming trinomial squares, page 108.

EXAMPLES

1. Solve $x^2 + 6x - 16 = 0.$ (1)

Solution: Transposing, $x^2 + 6x = 16.$ (2)

Adding 9 to each member of (2),

$$x^2 + 6x + 9 = 25. \quad (3)$$

Then $(x + 3)^2 = 5^2.$ (4)

Extracting the square root of each member of (4),

$$x + 3 = \pm 5. \quad (5)$$

Whence $x = -3 + 5$, or 2,

and $x = -3 - 5$, or -8.

Check: Substituting 2 for x in (1),

$$4 + 12 - 16 = 0,$$

or $0 = 0.$

Substituting -8 for x in (1),

$$64 - 48 - 16 = 0,$$

or $0 = 0.$

The 9 added to each member of (2) is the square of half the coefficient of x; that is, $9 = (\frac{6}{2})^2$, or 3^2. If the coefficient of x^2 is 1, the trinomial square can always be completed by **adding the square of half the coefficient of x.** If the coefficient of x^2 is -1 or any number other than $+1$, the equation is solved as in the next example.

The sign ± properly belongs to each member of (5). Thus $\pm(x + 3) = \pm 5$. This, however, gives precisely the same roots as $x + 3 = \pm 5$, a fact which the student can easily verify. For this reason the sign ± is put before one member only in equations obtained as was equation (5).

2. Solve $$3x^2 - 7x - 20 = 0. \tag{1}$$

Solution: Transposing, $$3x^2 - 7x = 20. \tag{2}$$

Dividing (2) by the coefficient of x^2,

$$x^2 - \frac{7x}{3} = \frac{20}{3}. \tag{3}$$

Adding $(\frac{7}{6})^2$ to each member of (3),

$$x^2 - \frac{7}{3}x + (\tfrac{7}{6})^2 = \frac{20}{3} + \frac{49}{36} = \frac{289}{36}. \tag{4}$$

Then $$(x - \tfrac{7}{6})^2 = (\tfrac{17}{6})^2. \tag{5}$$

Extracting the square root of each member of (5),

$$x - \tfrac{7}{6} = \pm \tfrac{17}{6}.$$

Whence $$x = \tfrac{7}{6} \pm \tfrac{17}{6}$$
$$= 4 \text{ or } -\tfrac{5}{3}.$$

Check: Substituting 4 for x in (1),

$$3 \cdot 4^2 - 7 \cdot 4 - 20 = 0.$$
$$48 - 28 - 20 = 0, \text{ or } 0 = 0.$$

Substituting $-\frac{5}{3}$ for x in (1),

$$3(-\tfrac{5}{3})^2 - 7(-\tfrac{5}{3}) - 20 = 0.$$
$$\tfrac{25}{3} + \tfrac{35}{3} - 20 = 0.$$
$$\tfrac{60}{3} - 20 = 0, \text{ or } 0 = 0.$$

3. Solve $$4x^2 - 4x - 79 = 0. \tag{1}$$

Solution: Transposing, $$4x^2 - 4x = 79. \tag{2}$$

Dividing each member of (2) by 4,

$$x^2 - x = \tfrac{79}{4}. \tag{3}$$

Adding $(\frac{1}{2})^2$ to each member of (3),

$$x^2 - x + (\tfrac{1}{2})^2 = \tfrac{79}{4} + \tfrac{1}{4} = \tfrac{80}{4}. \tag{4}$$

Then $$(x - \tfrac{1}{2})^2 = 20. \tag{5}$$

Extracting the square root of each member of (5),

$$x - \tfrac{1}{2} = \pm\sqrt{20}. \tag{6}$$

Whence $$x = \tfrac{1}{2} \pm 2\sqrt{5}. \tag{7}$$

Now $$\sqrt{5} = 2.2360+.$$

Then $$\tfrac{1}{2} + 2\sqrt{5} = .5 + 4.472+ = 4.972+. \tag{8}$$

Also $$\tfrac{1}{2} - 2\sqrt{5} = .5 - 4.472+ = -3.972+. \tag{9}$$

Check: Since (8) and (9) are not the *exact* values of x, they will, if substituted for x in (1), make its first member *nearly* but *not quite* zero. An exact check can be obtained by substituting the radical forms of the roots from equation (7) in equation (1).

The check may be shortened by substituting both roots at the same time as follows:

Substituting $\frac{1}{2} \pm 2\sqrt{5}$ for x in $4x^2 - 4x - 79 = 0$.

$$4\left(\tfrac{1}{2} \pm 2\sqrt{5}\right)^2 - 4\left(\tfrac{1}{2} \pm 2\sqrt{5}\right) - 79 = 0,$$
$$4\left(\tfrac{1}{4} \pm 2\sqrt{5} + 20\right) - 2 \mp 8\sqrt{5} - 79 = 0,$$
$$1 \pm 8\sqrt{5} + 80 - 2 \mp 8\sqrt{5} - 79 = 0.$$

The radical terms vanish because the two upper signs before them must first be taken together and then the two lower signs.

Therefore $1 + 80 - 2 - 79 = 0$, or $81 - 81 = 0$.

In quadratic equations like the preceding the radical forms of the roots are often sufficient; at other times values to two or three decimal places are necessary. Unless otherwise directed, obtain only the radical forms of irrational roots.

The student should note that the equations like $4x^2 - 4x - 79 = 0$ can be solved either graphically or by completing the square, but that their solution by factoring, though not impossible, is beyond him.

The method of solving a quadratic equation in x illustrated in the three examples preceding may be stated in the

Rule. *Transpose so that the terms containing* x *are in the first member and those which do not contain* x *are in the second.*

Divide both members of the equation by the coefficient of x^2 *(unless the coefficient of* x^2 *is* $+1$*).*

Then add to both members the square of one half the coefficient of x *(in the equation just obtained), thus making the first member a perfect trinomial square.*

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation and write the sign ± *before the square root of the second member, thus obtaining two linear equations.*

Solve for x *the equation in which the second member is taken with the sign* $+$ *and then solve the equation in which the second member is taken with the sign* $-$. *The results are the roots of the quadratic.*

Check. Substitute each root separately for x in the *original* equation. If the resulting equations are not obvious identities, simplify each until it becomes one.

EXERCISES

(Obtain the values of the radical answers in Exercises 10, 13, and 21 correct to three decimal places.)

Solve by completing the square, and check results:

1. $x^2 - 8x - 48 = 0.$

2. $x^2 - 5x - 14 = 0.$

3. $x(x + 2) - 5(x + 2) = 0.$

4. $\frac{3}{4} - y = y^2.$

5. $2y^2 - 9y + 4 = 0.$

6. $2y^2 + 5y = 0.$

7. $t^2 - 2t - 15 = 0.$

8. $3t^2 - 7t = 6.$

9. $9v = 5v^2 - 2.$

10. $x^2 - 2x - 4 = 0.$

11. $v^2 - \frac{7}{12}v - 1 = 0.$

12. $s^2 - 2s - 3\frac{1}{2} = 0.$

13. $h^2 + 10h + 13 = 0.$

14. $12t^2 - 25t + 12 = 0.$

15. $42 + 2z^2 = -19z.$

16. $15x^2 + 4x = 4.$

17. $1 - 6v^2 = 2v.$

18. $20s^2 + s = 1.$

19. $9x + 4 = 9x^2.$

20. $25x^2 - 20x - 12 = 0.$

21. $4x^2 = 1 - 4x.$

22. $x^2 + 4\sqrt{5}\,x = 25.$

23. $x^2 - 3\sqrt{2}\,x + 4 = 0.$

24. $1 - \frac{3\sqrt{3}}{2R} + \frac{3}{2R^2} = 0.$

25. $\frac{2x}{9} + \frac{2}{3} = \frac{7}{18x}.$

26. $\frac{1}{3} + \frac{x}{9} - \frac{2}{x} = 0.$

27. $\frac{2}{5v^2} + \frac{1}{4} - \frac{11}{10v} = 0.$

28. $\frac{3a}{2} + \frac{1}{2} - \frac{1}{3a} = 0.$

29. $\frac{25c}{2} - \frac{40}{c} - 40 = 0.$

30. $(x - 4)^2 - 3(x - 9) = 15.$

31. $(x - 2)(x + 3) = x(5x - 9) - 2.$

32. $x - \frac{3}{x + 2} = 0.$

33. $\frac{v^2}{v - 5} + \frac{5}{2} = 0.$

34. $\frac{3}{t - 7} + \frac{t}{4} = 0.$

35. $\frac{s}{s - 2} + \frac{s - 2}{s} = \frac{5}{2}.$

36. $\frac{7}{2y - 3} - \frac{5}{1 - y} = 12.$

37. $\frac{3 + x}{4 + x} - \frac{x - 5}{x - 6} = \frac{1}{12}.$

38. If $y = 2$, solve for x the equation $x^2 - xy - 3y^2 = -12.$

39. If $x = -3$, solve $x^2 - 4xy + x^3 + y^2 + 5 = 0$ for y.

40. $x^4 - 5x^2 + 4 = 0$.

This is not a quadratic equation, but many equations of this form can be solved by completing the square.

Solution:

$$x^4 - 5x^2 + 4 = 0.$$
$$x^4 - 5x^2 = -4.$$
$$x^4 - 5x^2 + \tfrac{25}{4} = -4 + \tfrac{25}{4} = \tfrac{9}{4}.$$
$$x^2 - \tfrac{5}{2} = \pm\tfrac{3}{2}.$$
$$x^2 = 4 \text{ or } 1.$$

Whence $x = \pm 2$ or ± 1.

Check as usual.

41. $x^4 - 13x^2 + 36 = 0$.

42. $4x^4 - 5x^2 + 1 = 0$.

43. $9x^4 - 37x^2 + 4 = 0$.

44. $4k^4 = 9k^2 - 2$.

45. $9s^4 + 12 = 31s^2$.

46. $4v^4 + 5 = 21v^2$.

47. Point out the error in the following:

$$9 - 30 = 49 - 70.$$
$$9 - 30 + 25 = 49 - 70 + 25.$$
$$(3-5)^2 = (7-5)^2.$$
$$3 - 5 = 7 - 5.$$
$$3 = 7.$$

101. Quadratic equations with literal coefficients. Such equations are solved as in Exercises 1 and 15 which follow.

EXERCISES

Solve for x by completing the square, and check:

1. $2a^2x^2 - ax - 1 = 0$.

Solution: Transposing, $2a^2x^2 - ax = 1$.

Dividing by $2a^2$, $x^2 - \dfrac{x}{2a} = \dfrac{1}{2a^2}$.

Completing the square, $x^2 - \dfrac{x}{2a} + \left(\dfrac{1}{4a}\right)^2 = \dfrac{1}{2a^2} + \dfrac{1}{16a^2}$.

Then $\left(x - \dfrac{1}{4a}\right)^2 = \dfrac{9}{16a^2}$.

Extracting the square root, $x - \dfrac{1}{4a} = \pm\dfrac{3}{4a}$.

EXERCISES

(Obtain the values of the radical answers in Exercises 10, 13, and 21 correct to three decimal places.)

Solve by completing the square, and check results:

1. $x^2 - 8x - 48 = 0.$
2. $x^2 - 5x - 14 = 0.$
3. $x(x + 2) - 5(x + 2) = 0.$
4. $\frac{3}{4} - y = y^2.$
5. $2y^2 - 9y + 4 = 0.$
6. $2y^2 + 5y = 0.$
7. $t^2 - 2t - 15 = 0.$
8. $3t^2 - 7t = 6.$
9. $9v = 5v^2 - 2.$
10. $x^2 - 2x - 4 = 0.$
11. $v^2 - \frac{7}{12}v - 1 = 0.$
12. $s^2 - 2s - 3\frac{1}{2} = 0.$
13. $h^2 + 10h + 13 = 0.$
14. $12t^2 - 25t + 12 = 0.$
15. $42 + 2z^2 = -19z.$
16. $15x^2 + 4x = 4.$
17. $1 - 6v^2 = 2v.$
18. $20s^2 + s = 1.$
19. $9x + 4 = 9x^2.$
20. $25x^2 - 20x - 12 = 0.$
21. $4x^2 = 1 - 4x.$
22. $x^2 + 4\sqrt{5}x = 25.$
23. $x^2 - 3\sqrt{2}x + 4 = 0.$
24. $1 - \frac{3\sqrt{3}}{2R} + \frac{3}{2R^2} = 0.$
25. $\frac{2x}{9} + \frac{2}{3} = \frac{7}{18x}.$
26. $\frac{1}{3} + \frac{x}{9} - \frac{2}{x} = 0.$
27. $\frac{2}{5v^2} + \frac{1}{4} - \frac{11}{10v} = 0.$
28. $\frac{3a}{2} + \frac{1}{2} - \frac{1}{3a} = 0.$
29. $\frac{25c}{2} - \frac{40}{c} - 40 = 0.$
30. $(x - 4)^2 - 3(x - 9) = 15.$
31. $(x - 2)(x + 3) = x(5x - 9) - 2.$
32. $x - \frac{3}{x + 2} = 0.$
33. $\frac{v^2}{v - 5} + \frac{5}{2} = 0.$
34. $\frac{3}{t - 7} + \frac{t}{4} = 0.$
35. $\frac{s}{s - 2} + \frac{s - 2}{s} = \frac{5}{2}.$
36. $\frac{7}{2y - 3} - \frac{5}{1 - y} = 12.$
37. $\frac{3 + x}{4 + x} - \frac{x - 5}{x - 6} = \frac{1}{12}.$

38. If $y = 2$, solve for x the equation $x^2 - xy - 3y^2 = -12.$

39. If $x = -3$, solve $x^2 - 4xy + x^3 + y^2 + 5 = 0$ for y.

40. $x^4 - 5x^2 + 4 = 0.$

This is not a quadratic equation, but many equations of this form can be solved by completing the square.

Solution:

$$x^4 - 5x^2 + 4 = 0.$$
$$x^4 - 5x^2 = -4.$$
$$x^4 - 5x^2 + \tfrac{25}{4} = -4 + \tfrac{25}{4} = \tfrac{9}{4}.$$
$$x^2 - \tfrac{5}{2} = \pm \tfrac{3}{2}.$$
$$x^2 = 4 \text{ or } 1.$$

Whence $x = \pm 2$ or ± 1.

Check as usual.

41. $x^4 - 13x^2 + 36 = 0.$

42. $4x^4 - 5x^2 + 1 = 0.$

43. $9x^4 - 37x^2 + 4 = 0.$

44. $4k^4 = 9k^2 - 2.$

45. $9s^4 + 12 = 31s^2.$

46. $4v^4 + 5 = 21v^2.$

47. Point out the error in the following:

$$9 - 30 = 49 - 70.$$
$$9 - 30 + 25 = 49 - 70 + 25.$$
$$(3-5)^2 = (7-5)^2.$$
$$3 - 5 = 7 - 5.$$
$$3 = 7.$$

101. Quadratic equations with literal coefficients. Such equations are solved as in Exercises 1 and 15 which follow.

EXERCISES

Solve for x by completing the square, and check:

1. $2a^2x^2 - ax - 1 = 0.$

Solution: Transposing, $2a^2x^2 - ax = 1.$

Dividing by $2a^2$, $x^2 - \dfrac{x}{2a} = \dfrac{1}{2a^2}.$

Completing the square, $x^2 - \dfrac{x}{2a} + \left(\dfrac{1}{4a}\right)^2 = \dfrac{1}{2a^2} + \dfrac{1}{16a^2}.$

Then $\left(x - \dfrac{1}{4a}\right)^2 = \dfrac{9}{16a^2}.$

Extracting the square root, $x - \dfrac{1}{4a} = \pm \dfrac{3}{4a}.$

Whence $$x = \frac{1}{a} \text{ or } -\frac{1}{2a}.$$

Check: Substituting $\frac{1}{a}$ for x in the original equation,

$$2a^2\left(\frac{1}{a}\right)^2 - a\left(\frac{1}{a}\right) - 1 = 0, \text{ or } 2 - 1 - 1 = 0.$$

Substituting $-\frac{1}{2a}$ for x in the original equation,

$$2a^2\left(-\frac{1}{2a}\right)^2 - a\left(-\frac{1}{2a}\right) - 1 = 0, \text{ or } \frac{1}{2} + \frac{1}{2} - 1 = 0.$$

2. $x^2 - ax - 2a^2 = 0.$

3. $x^2 + 2ax + a^2 - 4 = 0.$

4. $x^2 + 1 = a + 2x.$

5. $3x^2 - ax = 10a^2.$

6. $3mx + 2m^2 = 2x^2.$

7. $a^2x^2 - 7ax + 10 = 0.$

8. $4x^2 + 4ax - 3a^2 = 0.$

9. $x^2 + 4\sqrt{a}\,x - 5a = 0.$

10. $2x^2 + 9x\sqrt{h} = 5h.$

11. $a^2x^2 + 2ab = a^2 + b^2.$

12. $5x^2 + ax = x.$

13. $x(x - b) = a(a + b).$

14. $1 - \frac{4}{x} + \frac{4}{x^2} = \frac{b}{4x^2}.$

15. $x^2 + 2x = ax + 2a.$

HINT. $$x^2 + (2 - a)x = 2a.$$

$$x^2 + (2 - a)x + \left(\frac{2 - a}{2}\right)^2 = 2a + \frac{4 - 4a + a^2}{4}.$$

$$\left(x + \frac{2 - a}{2}\right)^2 = \frac{4 + 4a + a^2}{4}, \text{ etc.}$$

16. $x^2 - (a + 1)x + a = 0.$

17. $x^2 + bx + cx + bc = 0.$

18. $x^2 - ax + 4x - 4a = 0.$

19. $x^2 + 2a^2b^2 = a^2x + 2b^2x.$

20. $\frac{1}{a} - \frac{1}{5x} = 1 - \frac{5x}{a}.$

21. $x^2 + bx + c = 0.$

22. $ax^2 + bx + c = 0.$

102. Solution by formula. The equation

$$ax^2 + bx + c = 0$$

is the *general quadratic equation* in standard form. The student solved this equation in the preceding exercise and found

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad \text{(F)}$$

The value (F) is a general result and may be used as a formula to solve any quadratic equation. The solution of a quadratic by formula requires less labor than any other method, except for such equations as can be solved by factoring at sight. Those with considerable experience in algebra seldom solve a quadratic by any other method than by formula.

EXAMPLES

Solve by formula and check:

1. $3x^2 - 5x = 8$.

Solution: Writing in standard form, $3x^2 - 5x - 8 = 0$.

Then 3 corresponds to a, -5 to b, and -8 to c in the general quadratic $ax^2 + bx + c = 0$. Substituting these values in (F), where

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

gives

$$x = \frac{-(-5) \pm \sqrt{25 - 4 \cdot 3(-8)}}{2 \cdot 3}$$

$$= \frac{5 \pm \sqrt{25 + 96}}{6} = \frac{5 \pm 11}{6} = \frac{8}{3} \text{ or } -1.$$

Check as usual.

2. $2k^2x^2 = kx + 1$.

Solution: Writing in standard form, $2k^2x^2 - kx - 1 = 0$.

Then $a = 2k^2$, $b = -k$, and $c = -1$.

Substituting these values in the formula (F),

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4 \cdot 2k^2(-1)}}{2 \cdot 2k^2}.$$

$$x = \frac{k \pm \sqrt{k^2 + 8k^2}}{4k^2} = \frac{k \pm 3k}{4k^2} = \frac{1}{k} \text{ or } -\frac{1}{2k}.$$

Check: Substituting $\frac{1}{k}$ for x in the original equation,

$$2k^2\left(\frac{1}{k}\right)^2 = k\left(\frac{1}{k}\right) + 1, \text{ or } 2 = 1 + 1.$$

Substituting $-\frac{1}{2k}$ for x in the original equation,

$$2k^2\left(-\frac{1}{2k}\right)^2 = k\left(-\frac{1}{2k}\right) + 1, \text{ or } \frac{1}{2} = -\frac{1}{2} + 1.$$

EXERCISES

Solve for x by formula and check:

1. $2x^2 + 5x + 2 = 0.$
2. $3x^2 + 5x = 2.$
3. $x^2 - 3x - 10 = 0.$
4. $2x + 2 = x^2.$
5. $x^2 - x = 1.$
6. $2x^2 - \frac{11x}{2} - \frac{15}{2} = 0.$
7. $2x^2 - 3x = 1.$
8. $4x + 5 = x^2.$
9. $x^2 + x\sqrt{5} = 10.$
10. $12x = 1 - 72x^2.$
11. $x^2 + 2hx - 3h^2 = 0.$
12. $2m^2 = 9mx + 5x^2.$
13. $2x^2 + kx - 3k^2 = 0.$
14. $x^2 + 2x\sqrt{a} - 3a = 0.$
15. $mx = -m^2 + 6x^2.$
16. $x^2 + \frac{Kx}{2}\sqrt{2} - K^2 = 0.$
17. $n^2x^2 - 3knx - 10k^2 = 0.$
18. $6m^2x^2 + 19mnx = 7n^2.$
19. $x^2 + 2x = hx + 2h.$

HINT. $x^2 + (2-h)x - 2h = 0$. Then $a = 1$, $b = 2 - h$, and $c = -2h$. Substituting these values in (F),

$$x = \frac{-(2-h) \pm \sqrt{(2-h)^2 - 4 \cdot 1 \cdot (-2h)}}{2}, \text{ etc.}$$

20. $x^2 + rx - sx - rs = 0.$
21. $2x^2 + rs = rx + 2sx.$
22. $mnx^2 + nx = 3mx + 3.$
23. $mhx^2 + 4hx = 3mx + 12.$

PROBLEMS

(Reject all answers which do not satisfy the conditions of the problems.)

1. The sum of the square of a certain number and twice the number itself is 15. Find the number.

2. Find two numbers whose difference is 11 and whose product is 42.

3. If from twice the square of a certain number the number itself be taken away, the remainder is 45. Find the number.

4. Find two consecutive numbers whose product is 462.

5. Find two consecutive odd numbers whose product is 255.

6. Find three consecutive even numbers whose sum is $\frac{1}{6}$ of the product of the first two.

7. A rectangular field is 16 rods longer than it is wide. Its area is 32 acres (1 acre = 160 square rods). Find the dimensions of the field.

8. The sum of a certain number and its reciprocal is $\frac{41}{20}$. Find the number.

9. The area of a triangular field is $5\frac{5}{8}$ acres. The base is 51 rods longer than the altitude. Find the base and the altitude.

10. Two square fields together contain 62.5 acres. A side of one is 20 rods longer than a side of the other. Find the side of each.

11. The area of a rectangle is 18 square inches less than twice the area of a square. The rectangle is 7 inches longer than the square, and a side of the latter equals the breadth of the rectangle. Find the side of the square.

12. The hypotenuse of a right triangle is 41 feet. One leg is 31 feet shorter than the other. Find the legs.

13. The legs of a right triangle are in the ratio of $3:4$. The hypotenuse is 20. Find the legs.

Fact from Geometry. If one angle of a right triangle is 30 degrees, the hypotenuse is twice the shorter leg. Conversely, if the hypotenuse of a right triangle is double one leg, one angle of the triangle is 30 degrees.

14. One angle of a right triangle is 30 degrees and its longer leg is 9. Find, correct to two decimal places, the other two sides.

15. The hypotenuse of a right triangle is 10 and one leg is $5\sqrt{3}$. Show that one angle of the triangle is 30 degrees and find the number of degrees in each angle of the triangle.

16. The area of a square in square feet and its perimeter in linear feet are expressed by the same number. Find the side.

17. The area of a square in square feet and its perimeter in inches are expressed by the same number. Find the side.

18. The area of a square in square inches and its perimeter in feet are expressed by the same number. Find the side of the square.

19. The dimensions of a certain rectangle and the longest straight line which can be drawn on its surface are represented in inches by three consecutive even numbers. Find its dimensions.

20. The dimensions of a rectangular box are in the ratio of 1 : 2 : 3. Find the edges, if the entire outer surface is 792 square inches.

21. The edges of two cubical bins differ by one yard. Their volumes differ by 61 cubic yards. Find the edge of each bin.

22. The rates of two trains differ by 5 miles per hour. The faster requires 1 hour less time to run 280 miles. Find the rate of each train.

23. An automobile made a round trip of 160 miles in 9 hours. Returning, the rate was increased 4 miles per hour. Find the rate each way.

24. A page of a certain book is 2 inches longer than it is wide. The printed portion covers half the area of the page and the margin is 1 inch wide. Find the length and width of the page.

25. A man paid $16,000 for a farm. Later he sold all but 40 acres of it for the same sum, thereby gaining $20 on each acre sold. Find the number of acres in the farm.

26. The price of oranges being raised 10 cents per dozen, one gets 5 fewer oranges for 50 cents. Find the original price.

27. Two pumps together can fill a standpipe in 45 minutes. One pump alone requires 2 hours less time than the other. Find the time each requires alone.

28. Each of two trains ran 200 miles. One ran 7 miles per hour faster than the other and required 1 hour and 45 minutes less time. Find the rate of each train.

29. A and B leave point P at the same time, A going north and B east. Five hours later A has traveled 17 miles more than B and the distance between them is 53 miles. Find the rate of each.

30. A and B leave point P at the same time, A going northwest and B southwest. Five hours later A has traveled 9 miles less and B has traveled 8 miles less than the distance between them. Find the rate of each.

31. A stone, dropped from a balloon which was passing over a river, struck the water 15 seconds later. How high was the balloon at the time the stone was dropped?

HINT. The distance, S, through which a body falls from rest in t seconds is given by the equation $S = \frac{gt^2}{2}$, g being 32 feet.

32. A man drops a stone over a cliff and hears it strike the ground below $6\frac{1}{2}$ seconds later. If sound travels 1152 feet per second, find the height of the cliff.

GEOMETRICAL PROBLEMS

(The circumference of a circle is $2\pi R$, R being the radius. In the following problems use $\frac{22}{7}$ for π.)

1. The circumference of a carriage wheel is 11 feet. How many revolutions will it make while the carriage goes 55 yards?

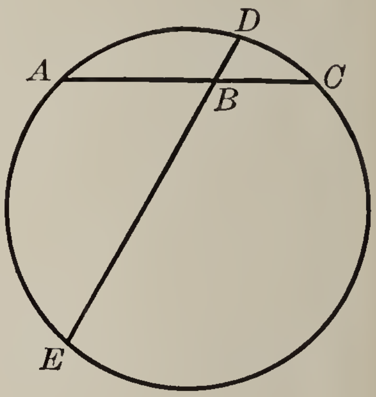
2. The radius of a carriage wheel is 2 feet. How many revolutions does the wheel make while the carriage goes 132 yards?

3. The circumference of a fore wheel of a carriage is 2 feet less than that of a rear wheel. In going 140 yards the smaller wheel makes 5 revolutions more than the larger. Find the circumference of each wheel.

4. In going 100 yards a fore wheel of a carriage makes 5 revolutions more than one of the rear wheels. The circumference of one wheel is 2 feet less than that of the other. Find the circumference and the radius of each wheel.

5. The circumferences of two wheels of a wagon differ by 2 feet. Together the two wheels make 11 revolutions while the wagon goes a distance of 20 yards. Find the diameter of each wheel.

6. The radii of two circles differ by 7 inches, and their areas differ by 770 square inches. Find their radii.



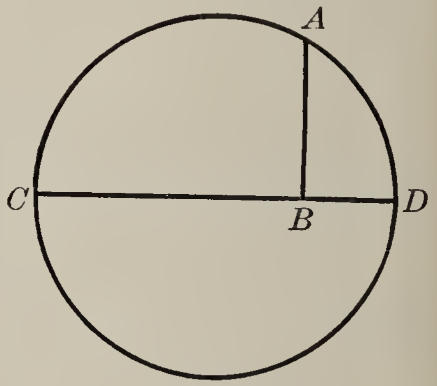
If any two chords of a circle, AC and DE, cross at B, then

$$AB \times BC = DB \times BE.$$

7. In the adjacent figure $AC = 18$, $DB = 4$, and $BE = 20$. Find AB.

8. In the adjacent figure $AC = 5$ feet, $DB = 18$ inches, and $BE = 48$ inches. Find AB and BC.

If AB is a line drawn from a point on the circumference perpendicular to the diameter, CD, of the circle, then $AB \cdot AB = CB \cdot BD$.



9. In the adjacent figure $AB = 9$ and $CD = 30$. Find CB and BD.

10. In the adjacent figure $AB = 20$ and $CB = 2\,BD$. Find BD.

11. A line AB is 12 inches long. A point P is located on AB so that $AB : AP = AP : PB$. Find the length of AP. (Can a meaning be given to both answers?)

History of the quadratic equation. Though the development of the method of solving quadratic equations is closely connected with the general growth of algebra, yet it is possible to indicate the most important steps in the process rather briefly.

The first writer on formal algebra was Diophantos, who lived at Alexandria, in Egypt, about 300 A.D. Most of his work that is preserved is devoted to the solution of problems that lead to equations. So far as we know he was the first to indicate the unknown number by a single letter, in this respect being far in advance of many mathematicians who lived much later. It is a little remarkable, in fact, that as able and original a man as Diophantos should have exerted so little influence on his successors. He solved his quadratic equations by a method not unlike that of completing the square, but his imperfect knowledge of the nature of numbers made it impossible for him to understand the entire significance of the process. Though he made every effort not to consider equations whose roots were not positive integers, sometimes they would creep in, and under such circumstances, when his method led him to a negative or irrational root, he rejected the whole equation as absurd or impossible. Even when both of the roots were positive he took only the one afforded by the positive sign in the formula for solving a quadratic.

The difficulties of Diophantos are typical of those encountered by mathematicians for the next fifteen hundred years. The difficulty lay, not in finding a formal method of solving the equation, but in understanding the result after it was obtained. The meanings of negative and of imaginary numbers have been two of the most difficult of all mathematical ideas for men to grasp.

Five or six hundred years later the Hindus devised a general solution of the quadratic, but their chief advance over Diophantos lay in the fact that they did not regard an equation whose roots were negative as necessarily absurd, but merely rejected the negative result of solving such an equation with the remark, "It is inadequate; people do not approve of negative roots." The Hindus, however, did realize that a quadratic equation sometimes has two roots, a fact that Diophantos never comprehended. They even went so far as to illustrate the difference between positive and negative numbers by assets and debts.

No material gain in the understanding of the solutions of the quadratic can be found until the seventeenth century. The keenest mathematicians of the sixteenth century, like Cardan and Vieta, rejected negative solutions regularly, though by this time irrational solutions were admitted. In fact, in 1544 Stifel, a German, published an algebra in which irrational numbers are included among the numbers proper. But he affirms that except in the case where a quadratic equation has two positive roots, no equation has more than one root. It was not until the work of Descartes and Gauss became widely known that the nature of the roots of all kinds of quadratic equations was completely understood.

CHAPTER XXIV

FUNDAMENTAL OPERATIONS

(In Part Review)

103. Order of fundamental operations. The numerical value of an expression which involves the signs of the four fundamental operations depends on the order in which the indicated operations are performed. The assumptions on page 10 may be stated as follows:

In a series of operations involving addition, subtraction, multiplication, and division, first the multiplications and divisions shall be performed in the order in which they occur. Then the additions and subtractions shall be performed in the order in which they occur or in any other order.

Within any parenthesis the preceding rule applies.

EXERCISES

Simplify:

1. $3 - 5 + 6 - 8.$ **3.** $24 \div 8 \cdot 4 - 4 + 6.$

2. $6 \div 2 + 1 - 4.$ **4.** $(7 - 6)(18 - 2 \cdot 4) \div (20 \div 4).$

5. $42 - 2(18 - 2 \cdot 3) \div 4 + 3 \cdot 5.$

6. $16 + 4 \div 8 - 10 + 51 \div 16 - 4 - 6 \cdot 3 \cdot 0 \cdot 2 + 18 \cdot 8 \div 48 - 2 \cdot 18 \div 12.$

7. $(16 \div 32 \times 48 \div 8 - 4 - 8 + 3) \times [12 \div 4 \div 3 - 1] + (42 \div 6 \cdot 7 - 42 - 6) \cdot 6.$

8. Does $a^4 = 4a$ when $a = 3$? when $a = 2$? when $a = 0$?

9. What name is given to each 4 in $a^4 = 4a$? Define each.

10. Define power. Distinguish between exponent and power.

Find the numerical value of:

11. $x^2 - 5x + 6$ when $x = 5$.

12. $x^3 - 3x^2 + 3x - 1$ when $x = 3$.

13. $x^3 - 3x^2y + 3xy^2 - y^3$ when $x = 4$ and $y = 2$.

14. $\dfrac{x^4 + x^2y^2 + y^4}{x^2 - xy + y^2} - \dfrac{x^3 + y^3}{x + y}$ when $x = 3$ and $y = 2$.

15. What is the absolute value of a number? Illustrate.

104. Addition and subtraction. The order in which a series of positive and negative numbers is written does not affect the algebraic sum. This principle is called the Commutative Law of Addition.

EXERCISES

1. Review the definition of term, of similar and dissimilar terms, and the rules on pages 17 and 30.

2. Review the rule for subtraction on page 40, and the rules and examples on pages 55–56.

Add:

3. 16, -3, $+2$, -8, -7, and 4.

4. $4a$, $-6a$, $-10a$, $+2a$, and $18a$.

5. $7x - 4y - z$, $3x + z - 8y$, and $18y - 17x - 14z$.

6. $4a^2 - 3a^2c - 4ac^2$, $3a^2c - 8ac^2 - 8a^2$, and $3a^2 - 6a^2c$.

7. If $x = 1$, $y = 2$, and $z = 3$, find the numerical value of each of the three expressions and of the result obtained in Exercise 5. Compare the sum of the three numerical values with the numerical value of the result.

8. State a rule for checking work in addition of algebraic expressions.

Write with polynomial coefficients:

9. $ay + by + cy$.

10. $3ax - 4bx + 6x$.

11. $4x - abx - x$.

12. $3(a + b) - c(a + b)$.

13. $4b(3x - 2) - 8c(3x - 2)$.

14. $4m(5a - 3c) - 6n(-3c + 5a)$

Subtract the first expression from the second in:

15. $4a$, $6a$.

16. $8a^3$, $5a^3$.

17. $4x + 3$, $8x + 6$.

18. $7x^2 - 10$, $5x^2 + 20$.

19. $x - 3y^2 + z - 4ac + 7ax$, $4x - y^2 + 8 - 5ax + 9ac$.

20. $a^3 - c + 3x - a^2m - 8ac$, $4a^3 + m - 8x - 10ac + 4a^2m$.

Find the expression which added to the first will give the second in:

21. $x^2 - 5x + 6$, $3x^2 - 5x + 2$.

22. $4x^2 - 3cx + c^2$, $8c^2 + 7cx - 10x^2 + 8$.

Find the expression which subtracted from the first will give the second in:

23. $4a^2 - 2ab + b^2$, $7a^2 - 10ab + 6b^2$.

24. $c^2 - 10cx + 8x^2$, $9x^2 - 10cx + 4 + c^2$.

25. State a rule for checking work in subtraction suggested by the directions preceding Exercises 21 and 23.

26. From the sum of $ax - ac - 3c^2$ and $4c^2 - 3ac$ take the sum of $4c^2 - 8ax + a^2$ and $4ac + 3ax - 5c^2$.

Remove parentheses and combine like terms:

27. $4x - 3 - (a - 2x) + (3x - a)$.

28. $6x + (3c - 8x + 2) - (c - x - 2)$.

29. $6x - [-(a - c) + (3c - 4a)]$.

30. $7c - [(3c - 4) - 6 - (4x - 3a - c)]$.

31. $6x - 4(3 - 5x) - 4[2(x - 4) + 3(2x - 1) - (x - 7)]$.

32. $3x - 2[1 - 3(2x - 3 - a) - 5\{a - (3x - 2a) - 4\}]$.

33. State the rule for the removal of a parenthesis:

(*a*) when it is preceded by the sign plus;

(*b*) when it is preceded by the sign minus.

Inclose in a parenthesis preceded by the sign plus those terms which contain x and y, and inclose all other terms in a parenthesis preceded by the sign minus.

34. $x^2 + 2xy + y^2 - a^2$. **35.** $x^2 + 14ab - 49a^2 - b^2$.

36. $y^2 + 6xy + 9x^2 - m^2 - 10m - 25$.

37. $x^4 + 10x^2y^3 - c^8 + 12c^4d - 36d^2 + 25y^6$.

38. State the rules for inclosing terms in a parenthesis preceded by (a) the sign plus; (b) the sign minus.

105. Multiplication and division. Meaning of a zero exponent. The laws for exponents stated in the formulas of § 27, page 60, and § 37, page 75, are assumed to hold for all values of a and b.

Then $$x^4 \div x^4 = x^{4-4} = x^0.$$

But $$x^4 \div x^4 = \frac{x^4}{x^4} = 1.$$

Therefore $$x^0 = 1.$$

More generally, $$x^a \div x^a = x^{a-a} = x^0,$$

and $$x^a \div x^a = \frac{x^a}{x^a} = 1.$$

As before, $$\boldsymbol{x^0 = 1}.$$

That is, any number (except zero) whose exponent is zero is equal to 1. Hence, if x is not zero, $4^0 = (\frac{2}{3})^0 = (-6)^0 = (5x)^0 = (x^2 - 2x + 1)^0$, for each equals 1.

106. Meaning of a negative exponent. If in the formula of § 37, page 75, b is greater than a, we obtain a negative exponent for n. The meaning of such an exponent is illustrated as follows:

$$a^3 \div a^5 = a^{3-5} = a^{-2}.$$

Obviously $$a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1}{a^2}.$$

Therefore a^{-2} is another way of writing $\frac{1}{a^2}$.

Then $$4^{-3} = \frac{1}{4^3} = \frac{1}{64}.$$

In general terms, $$\boldsymbol{x^{-a} = \frac{1}{x^a}}.$$

Consequently $$\frac{1}{x^{-a}} = \frac{1}{\frac{1}{x^a}} = x^a.$$

Similarly, we obtain the more general results

$$bx^{-a} = \frac{b}{x^a} \text{ and } \frac{b}{x^{-a}} = bx^a.$$

Hereafter it will be assumed that all the preceding exponential laws hold for positive, zero, and fractional exponents.

EXERCISES

1. Review the definitions on pages 59–64, the principle on page 60, and the rules on pages 60, 61, and 62.

Perform the indicated operation:

2. $(4x^2 - 3x)(2x)$. **3.** $(2x + 3)(5x - 6)$.

4. Substitute 2 for x in each of the factors of Exercise 2, and in the product. Compare the numerical value of the product with the product of the numerical values of the factors. Then state a method of checking numerically work in multiplication:

5. $(3x^2 - 1)^2$. **8.** $(e^x + 2e^{-x})^2$.

6. $(7x^{2a} - 8x^a + 3)^2$. **9.** $(e^x - e^{-x})^3$.

7. $(x^{\frac{1}{2}} + x^{\frac{1}{3}})^2$. **10.** $(e^{2x} - 3e^{-x})^4$.

11. $(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1)(x^{\frac{1}{2}} - x^{\frac{1}{4}} + 1)$.

12. $(4x^{3c} - 6x^c + 3)(7x^{3c} - x^{2c} + 4)$.

13. $(x^2 - 2xy^2 + y^4)(x^2 + 2xy^2 + y^4)$.

14. $(x^{-1} - 3x - 2x^{-2})^2$.

15. $(x^{-\frac{1}{2}} + 2x^{\frac{1}{2}} - 3x^{\frac{3}{2}})^2$.

16. $\left(\frac{2a^2}{3} - \frac{a}{5} + \frac{2}{7}\right)\left(\frac{2a^3}{3} + \frac{a^2}{5} - \frac{2a}{7}\right)$.

17. $(5x^{2a} - 3x^{-2a} - 6x^{-a} + 3x^a)^2$.

18. $x^2 - x - 90 = ?$ if $x = -9$.

19. $x^3 - 3x^2y + 3xy^2 - y^3 = ?$ if $x = 2$ and $y = -3$.

20. $x^3 + 3x^2y + 3xy^2 + y^3 = ?$ if $x = -4$ and $y = -2$.

21. State the Associative Law of Multiplication. Illustrate.

22. State the Distributive Law of Multiplication. Illustrate.

23. Review the principle on page 75 and the rules on pages 75–76 and 78–79.

24. $(8x^4 - 6x^2 - 4x) \div (-2x)$.

25. $(x^2 - 7x + 12) \div (x - 3)$.

26. $(e^1 - e^{-1})^2 = ?$ if $e = 2$; if $e = -3$.

27. $e^{2x} - 2e^0 + e^{-2x} = ?$ if $e = 2$ and $x = 2$.

28. $(x^3 - 64) \div (x - 4)(x^2 - 4x + 16)$.

29. $(x^4 - 8x^2 + 33x - 30) \div (x^2 + 3x - 5)$.

30. State a method of checking work in division similar to the check of multiplication.

Find the remainder in:

31. $(8x^3 - x^2 - 5) \div (2x - 3)$.

32. $(4x^4 - x^2 - 3) \div (2x^2 - x - 1)$.

Divide:

33. $x^3 + 8y^3 + 125 - 30xy$ by $x + 2y + 5$.

34. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

35. $3x^{-10} + x^6 - 4x^{-6}$ by $2x^{-2} + x^2 + 3x^{-6}$.

36. $x^{\frac{3}{5}} - y^{\frac{3}{5}}$ by $[(x^{\frac{1}{5}} - y^{\frac{1}{5}}) \div (x^{\frac{1}{10}} + y^{\frac{1}{10}})]$.

37. $9m + 4m^{-1} - 13$ by $3m^{\frac{1}{2}} - 5 + 2m^{-\frac{1}{2}}$.

38. $x^{2a} + 4x^{-2a} - 29$ by $x^a - 2x^{-a} - 5$.

39. $9x^{2a} + 25x^{-4a} - 19x^{-a}$ by $5x^{-2a} + 3x^a - 7x^{-\frac{a}{2}}$.

40. $\left(6a^3 + 6x^3 + \frac{35ax^2}{2} + \frac{35a^2x}{3}\right) \div \left(\frac{3a}{2} + \frac{2x}{3}\right)$.

41. $$\left(\frac{9a^5}{5} + \frac{243a^3}{20} - 12 + \frac{59a}{4} - \frac{443a^2}{30} - \frac{43a^4}{8}\right) \div \left(\frac{3a^3}{4} + a - \frac{3}{2} - \frac{5a^2}{6}\right).$$

107. Detached coefficients. An algebraic expression is rational and integral if its terms are rational and integral. Such an expression is usually called a **polynomial.**

An integral expression may not be rational. Nor is every rational expression integral. Thus, $\frac{x^2}{4} + \sqrt{x} + 8$ is integral but not rational, while $\frac{x^2}{4} + \frac{1}{x} + 8$ is rational but not integral.

A rational integral expression is **homogeneous** if its terms are all of the same degree.

Thus, $x^3 - x^2y - 7y^3$ and $3x^4 + x^2y^2 - y^4$ are homogeneous expressions of the third and the fourth degrees respectively.

In the multiplication or division of polynomials which involve but one letter or which are homogeneous in two letters much labor can be saved by using the coefficients only.

EXAMPLES

1. Multiply $3x^3 - 4x + 6$ by $2x^2 - 5x + 3$.

Solution: Since x^2 is missing in the first expression, its coefficient is zero. Inserting $0x^2$ and detaching coefficients, the multiplication is as follows:

$$\begin{array}{rrrrrr}
3 & +\ 0 & -\ 4 & +\ \ 6 & & \\
2 & -\ 5 & +\ 3 & & & \\
\hline
6 & +\ 0 & -\ 8 & +\ 12 & & \\
 & -15 & +\ 0 & +\ 20 & -\ 30 & \\
 & & +\ 9 & +\ \ 0 & -\ 12 & +\ 18 \\
\hline
6 & -15 & +\ 1 & +\ 32 & -\ 42 & +\ 18
\end{array}$$

Supplying the powers of x, we obtain as the product $6x^5 - 15x^4 + x^3 + 32x^2 - 42x + 18$.

2. Divide

$6x^4 - 11x^3y + 2x^2y^2 + 27xy^3 - 18y^4$ by $2x^2 - 5xy + 6y^2$.

Solution:

$$\begin{array}{rrrrr|l}
6 & -\ 11 & +\ \ 2 & +\ 27 & -\ 18 & 2 - 5 + 6 \\
\cline{6-6}
6 & -\ 15 & +\ 18 & & & 3 + 2 - 3 \\
\cline{1-3}
 & 4 & -\ 16 & +\ 27 & & \\
 & 4 & -\ 10 & +\ 12 & & \\
\cline{2-4}
 & & -\ \ 6 & +\ 15 & -\ 18 & \\
 & & -\ \ 6 & +\ 15 & -\ 18 & \\
\cline{3-5}
\end{array}$$

Therefore the quotient is $3x^2 + 2xy - 3y^2$.

In both multiplication and division by detached coefficients zero must be supplied for the coefficient of any missing term.

EXERCISES

Use detached coefficients and perform the indicated operation:

1. $(x^2 - 8x + 16)(2x - 3)$.

2. $(x^2 - 4x + 4)(x^2 + 4x + 4)$.

3. $(a^2 - ab + b^2)(a^2 + ab + b^2)$.

4. $(2x^2 + 5x + 2) \div (2x + 1)$.

5. $(x^3 + 4x - 16) \div (x - 2)$.

6. $(3xy - 6y^2 - 2x^2)(8x^2 - 6y^2 - 5xy)$.

7. $(9x^4 - 4x + 13x^2 + 4 - 6x^3) \div (3x^2 - x + 2)$.

8. $(x^4 + 4y^4) \div (x^2 - 2xy + 2y^2)$.

9. $(81a^4 - 171a^2b^2 + 25b^4) \div (9a^2 - 5b^2 + 9ab)$.

10. $(4a^3 - 2a^2 - 3a^{-2} - 5a^{-1} + 2a) \div (2a^2 - 2 - a^{-1})$.

11. $(8x - 12x^{\frac{2}{3}}y^{-1} + 6x^{\frac{1}{3}}y^{-2} - y^{-3}) \div (2x^{\frac{1}{3}} - y^{-1})$.

12. Which expressions in the preceding exercises are (a) not integral? (b) not rational?

Note. It is interesting to observe that our ordinary decimal notation really involves the use of detached coefficients. The number 649, for instance, is an abbreviated way of writing $6 \cdot 10^2 + 4 \cdot 10 + 9$. In fact, the various digits in any number in the decimal form are the detached coefficients of some power of the number 10.

108. Important special products. Certain products are of frequent occurrence. These should be memorized so that one can write or state the result without the labor of actual multiplication.

ORAL EXERCISES

1. Review the formulas, rules, and examples of pages 93–100.

Perform the indicated operation:

2. $(x + 3)^2$.

3. $(x - 5)^2$.

4. $(2x + 4)^2$.

5. $(4x - 3)^2$.

6. $(x^2 - x)^2$.

7. $(x - 3c^2)^2$.

8. $(3x^2 - 4xc)^2$.

9. $(x^2 - x^{-2})^2$.

10. $(x^4 - 3x^{-4})^2$.

11. $(16x^2 + 8x^2c + x^2c^2) \div (4x + xc) = ?$ Why?

12. $(2a^x - a^{-x})(2a^x - a^{-x})(2a^x - a^{-x}) \div (2a^x - a^{-x})$.

13. $(x - c)(x + c)$.

14. $(x + 6)(x - 6)$.

15. $(a - 3c)(a + 3c)$.

16. $(m - x)(x + m)$.

17. $(4x - 3c)(3c + 4x)$.

18. $(x^3 - cx)(x^3 + cx)$.

19. $(4c^3 - a^5)(a^5 + 4c^3)$.

20. $(x + 3)(x + 4)$.

21. $(b - 3)(b - 4)$.

22. $(c - 1)(c + 2)$.

23. $(x - 3)(x + 5)$.

24. $(a^2 - 4a)(a^2 + 6a)$.

25. $(a^x - 2a^{-x})(a^x + 5a^{-x})$.

26. $(cx - 4c^2)(cx + 8c^2)$.

27. $(a + b + c)^2$.

28. $(a + c + x)^2$.

29. $(a - c + x)^2$.

30. $(a - c - x)^2$.

31. $(a - c + 2)^2$.

32. $(x - c - 3a)^2$.

33. $\dfrac{a^2 + 9c^2 + x^2 - 6ac + 2ax - 6cx}{a - 3c + x} = ?$ Why?

34. $(3c - 5a + 2x)^2$.

35. $(x^{2a} + x^a - 5)^2$.

36. $(a + a^{-1} - 3)^2$.

37. $(a^x - a^{-x} + 4)^2$.

38. Can the expression in Exercise 27 be squared as a binomial? Explain.

39. $(x + c)^3$.

40. $(x - c)^3$.

41. $(x - 1)^3$.

42. $(x + 2)^3$.

43. $(x + 3)^3$.

44. $(x - 4)^3$.

45. $(x^2 - 2)^3$.

46. $(5a - 4c)^3$.

47. $(2x^5 - 7x^2)^3$.

48. $(x^3 - 6x^2 + 12x - 8) \div (x - 2) = ?$ Why?

49. $(8 - 12x + 6x^2 - x^3) \div (4 - 4x + x^2) = ?$ Why?

50. Square $(5 - 7)$ as a binomial and check the result by subtracting 7 from 5 and squaring the difference obtained.

51. Square $x + 9$ and $-x - 9$. Compare results and explain.

52. Find the product of $(9 - 4)(9 + 3)$ by the formula. Verify by simplifying each binomial and then multiplying.

53. Square by an appropriate algebraic formula: (*a*) 42, (*b*) 59, (*c*) 73, (*d*) 105, (*e*) 97, and (*f*) 1005.

54. Expand $(4 + 9 - 5)^2$ by the formula. Verify by simplifying and then squaring.

55. Expand $(3 - 2)^3$ by the formula. Verify by simplifying the binomial and then cubing the result.

56. Expand $(a-2b)^3$ and $(2b-a)^3$. Compare results and explain.

57. What must be added to $9x^2+6x$ to complete the trinomial square? (See page 108.)

58. What must be added to or subtracted from $16a^2+9$ to complete the trinomial square? Why?

Form a perfect trinomial square of:

59. $x^2 - ? + 9$.

60. $4x^2 - ? + 9a^2$.

61. $x^2 + 4x + ?$

62. $9x^2 \pm 24x + ?$

63. $? \pm 12x + 9$.

64. $9x^2 - 4ax + ?$

65. $a^2 + ? + a^{-2}$.

66. $a^{2x} - ? + a^{-2x}$.

67. $a^{2x} - ? + 16a^{-2x}$.

68. $a^{4x} + 10 + ?$

69. $a^{10} - ? + 49a^{-6}$.

70. $a^6 - 6a^2 + ?$

71. $a^{4x} - 12a^x + ?$

72. $4a^{6x} + ? + 25a^{-2x}$.

CHAPTER XXV

FACTORING AND FRACTIONS

(In Part Review)

109. Definitions. Factoring is the process of finding two or more prime expressions whose product is a given expression.

An integral expression is here regarded as prime when no two rational integral expressions can be found (excepting the expression itself and 1) whose product is the given integral expression.

The methods of this chapter enable one to factor all the more usual rational integral expressions in one letter which are not prime, as well as many of the simpler expressions in two letters. No attempt will be made even to define what is meant by prime factors of expressions which are not rational and integral.

110. Common factors, perfect trinomial squares, and quadratic trinomials. The following exercises review the work of §§ 49–52. The student should turn to those sections and make any necessary study of the type forms and the illustrative examples.

EXERCISES

Factor:

1. $4x + 8$. **3.** $a^2c - ac^2 - 4ac$. **5.** $x^{2a} - 3x^a + 12x$.

2. $ax - 7ay$. **4.** $3xy + 21y^5 - 15y^3$. **6.** $y^{2a} - 6y^a + 2y^{a-1}$.

7. $5xy + 30y(x^2 + xy)$.

8. $(7a^2 - 21ab + 7a) - 14ax$.

9. $(3c^2 - 3cd) - a(45c^2 - 15c^3x)$.

10. $2r^{2x+3} + 12r^{x-7} - 16sr^{x+2} + 8sr^{x+4}$.

Separate into polynomial factors:

11. $3(x+y)+a(x+y)$.

12. $a(x-3)-b(x-3)$.

13. $4x-4y+bx-by$.

HINT. $4x-4y+bx-by=4(x-y)+b(x-y)$, etc.

14. $3cx+6ac+8ax+4x^2$.

15. $-6x^2+10x+21xm-35m$.

16. $rs-2s+3r-6-5rx+10x$.

17. $x^{3a}-3x^{2a}-x^a+3-6x^{4a}+2x^{5a}$.

18. $x^{3a-2}+2x^{a+1}-15x^{2a-3}-10+10x^{2a-3}$.

19. x^2+6x+9.

20. $x^2+x+\frac{1}{4}$.

21. $9b^2-12b+4$.

22. $9r^2+49-42r$.

23. a^2-2+a^{-2}.

24. a^4-6+9a^{-4}.

25. $\dfrac{x^2}{9a^2}-\dfrac{8x}{a}+144$.

26. $x^{4a}-10ax^{2a}+25a^2$.

27. $r^{4x+2}+4-4r^{2x+1}$.

28. $a^{2x}+4a^{-2x}-4$.

29. $a^{4x}-2a^x+a^{-2x}$.

30. $4(a+5)^2-12b(a+5)+9b^2$.

31. $(a-b)^{2x}-18x(a-b)^x+81x^2$.

32. $x^2+2x-24$.

33. $x^2+x-\frac{3}{4}$.

34. $a^2+.3a-.1$.

35. $c^2-ac-90a^2$.

36. $r^2s^2+6rs-40$.

37. a^2+5+6a^{-2}.

38. $15m^2-14mx-x^2$.

HINT. This may be written as $-1(x^2+14xm-15m^2)$, etc.

39. $90+x-x^2$.

40. $12a^4-a^2x-x^2$.

41. $a^{2x}-20+19a^x$.

42. $a^2+12a^{-2}+7$.

43. $a^{2x}-8a^{-2x}-2$.

44. $120+7m^n-m^{2n}$.

45. $a^{4x}-a^x-6a^{-2x}$.

46. $(m+n)^{4e}-9(m+n)^{2e}-22$.

111. The general quadratic trinomial. The type form is

$$ax^2+bx+c.$$

This important type really includes the two preceding types. If a trinomial of this type has two *rational* factors, they have the forms $dx+e$ and $fx+g$.

Now $(dx + e)(fx + g) = dfx^2 + fex + dgx + ge \qquad (1)$

$= dfx^2 + (fe + dg)x + ge. \qquad (2)$

In (2) the product of the coefficient of x^2, df, and the constant term, ge, is $dfge$. But $dfge$ equals fe times dg, and fe plus dg equals the coefficient of x. Therefore, if $ax^2 + bx + c$ has rational factors, it can be written in the form (1) and factored by grouping terms. Hence for factoring expressions of the type $\boldsymbol{ax^2 + bx + c}$ we have the

RULE. *Find two numbers whose algebraic product is* **ac** *and whose algebraic sum is* ***b***.

Replace **bx** *by two terms in* **x** *whose respective coefficients are the numbers just found, and factor by grouping terms.*

EXERCISES

Separate into binomial factors:

1. $4x^2 - 7x - 15$.

Solution: Here $ac = 4 \cdot (-15)$, or -60, and $b = -7$. The numbers whose product is -60 and whose sum is -7 are $+5$ and -12.

Hence $4x^2 - 7x - 15 = 4x^2 - 12x + 5x - 15$

$= 4x(x - 3) + 5(x - 3) = (x - 3)(4x + 5)$.

2. $2a^2 - 3a - 2$.

3. $3a^2 + 8a - 3$.

4. $4a^2 + a - 5$.

5. $9c^2 - 71c - 8$.

6. $5r^2 - 22r + 8$.

7. $7x^2 + 62x - 9$.

8. $6x^2 + 19x - 7$.

9. $6x^2 + 13x - 5$.

10. $2x^2 + 7x - 15$.

11. $3x^2 - ax - 2a^2$.

12. $4a^4 - 12a^2 + 9$.

13. $25 + 4c^2d^2 - 20cd$.

14. $-8n^4 + 3n^8 - 3$.

15. $6x^{2y} - 13x^y + 6$.

16. $20x^2 - 9xy^3 - 20y^6$.

17. $c^2x^a + x^{2a} - 12c^4$.

18. $2x^2 - (a + 2b)x + ab$.

19. $5m^{2n-4} + 9am^{n-2} - 2a^2$.

20. $6x^{4a} + (3 - 2y^b)x^{2a} - y^b$.

21. $20a^2b^{4-2x} - 9a - 20b^{2x-4}$.

112. A binomial the difference of two squares. Review the type form and rule on page 114 and the examples on pages 115–116.

EXERCISES

Factor:

1. $m^2 - n^2$.
2. $x^2 - 4$.
3. $x^2 - \frac{1}{9}$.
4. $25x^2 - 49b^2$.
5. $a^2 - a^{-2}$.
6. $a^4 - \frac{4c^2}{25}$.
7. $16a^6 - 25c^4d^{10}$.
8. $a^{2x} - a^{-2x}$.
9. $a^4 - a^{-4}$.
10. $(a+c)^2 - 1$.
11. $(a-x)^4 - 4$.
12. $9 - (2+x)^2$.
13. $16 - (x-a)^{6a}$.
14. $5^{2a} - (m-1)^{10a}$.
15. $(a+c)^2 - (m-n)^2$.
16. $(a-x)^{2r} - (c-5)^2$.
17. $a^2 + 2ax + x^2 - 9$.

Hint. $a^2 + 2ax + x^2 - 9$
$= (a+x)^2 - 9$.

18. $25 - 10x + x^2 - 16m^2$.
19. $9 - 12a + 4a^2 - b^{2m}$.
20. $m^2 - a^2 - 6a - 9$.

Hint. $m^2 - a^2 - 6a - 9 = m^2 - (a^2 + 6a + 9) = m^2 - (a+3)^2$.

21. $x^2 - 4y^2 + 20y - 25$.
22. $-28c^3d^2 - 49c^6 + 1 - 4d^4$.
23. $12r^s - 36 + 5^{2a} - r^{2s}$.
24. $(m-2)^2 - 4n^2 + 28n - 49$.
25. $x^2 - 6x + 9 - y^2 + 8ay - 16a^2$.
26. $4bd + 4c^2 - 4d^2 - 4c - b^2 + 1$.
27. $6c + h^2 - 1 - 9c^2 - 4hk + 4k^2$.
28. $h^2 - 4y^2 - 10h + 8xy + 25 - 4x^2$.
29. $25x^2 - 20x + 4 - 4y^2 - 9a^2 - 12ay$.

113. Expressions reducible to the difference of two squares. The type form is

$$a^4 + ka^2b^2 + b^4.$$

If k has such a value that the trinomial is not a perfect square, a trinomial of this type can often be written as the difference of two squares. Thus, if $k = 1$, the adding and subtracting of a^2b^2 transforms the expression into the difference of two squares.

Example 1. Factor $a^4 + a^2b^2 + b^4$.

Solution:
$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab). \end{aligned}$$

Example 2. Factor $49h^4 + 34h^2k^2 + 25k^4$.

Solution: If $36h^2k^2$ is added, the expression becomes a perfect trinomial square. Adding and subtracting $36h^2k^2$, we have

$$\begin{aligned} 49h^4 + 34h^2k^2 + 25k^4 &= 49h^4 + 70h^2k^2 + 25k^4 - 36h^2k^2 \\ &= (7h^2 + 5k^2)^2 - (6hk)^2 \\ &= (7h^2 + 5k^2 + 6hk)(7h^2 + 5k^2 - 6hk). \end{aligned}$$

EXERCISES

Factor:

1. $x^4 + x^2 + 1$.
2. $x^4 + x^2y^2 + y^4$.
3. $x^4 + 4x^2 + 16$.
4. $16y^4 + 4y^2 + 1$.
5. $c^4 + c^2d^2 + 25d^4$.
6. $1 - 19y^6 + 25y^{12}$.
7. $4x^4 + 3x^2y^2 + 9y^4$.
8. $4x^4 - 28x^2y^6 + 9y^{12}$.
9. $9a^8 - 19a^4b^2 + 25b^4$.
10. $49h^4 - 44h^2k^4 + 4k^8$.
11. $x^8 + x^4 + 1$.
12. $c^8 - 6c^4 + 1$.
13. $16 + 4x^4 + x^8$.
14. $y^8 + 16y^4 + 256$.
15. $3x^9y + 3x^5y^5 + 3xy^9$.
16. $16h^4 - 33h^2k^2 + 36k^4$.
17. $25c^4 - 51c^2d^2 + 49d^4$.
18. $49a^4 - 32a^2b^2 + 64b^4$.
19. $64x^4 + 119x^2y^2 + 81y^4$.
20. $81a^4 - 171a^2b^2 + 25b^4$.
21. $1 + 4x^4$. HINT. $1 + 4x^4 = 1 + 4x^2 + 4x^4 - 4x^2$.
22. $64c^4 + 1$.
23. $x^4 + 4y^4$.
24. $x^8 + 4y^8$.
25. $x^8 + 64$.
26. $x^{4a} + 4y^{8a}$.
27. $a^{4b}c^{12d} + 64e^{4x+4}$.

114. The sum or difference of two cubes. Review the type form and the examples on page 118.

EXERCISES

Factor:

1. $x^3 + 64$.
2. $x^3 + 27$.
3. $a^3 - 64$.
4. $8 + m^3$.
5. $27 - m^6$.
6. $x^3 - \frac{1}{8}a^3$.
7. $x^3 - y^{-3}$.
8. $8a^3 + 27b^3$.
9. $1 - 125x^6$.
10. $\frac{1}{27}x^3 - y^6$.
11. $x^6 - y^6$.
12. $x^6 + y^6$.
13. $x^9 - a^3$.
14. $x^6 + a^9$.
15. $(x + y)^3 - 8$.

16. $x^3 - 9x^2 + 27x - 28.$

17. $a^3b^6c^9 - 8d^{12}.$

18. $x^{3c} - y^{-3c}.$

19. $c^{6c} + 27d^{9x}.$

115. The Remainder Theorem. If any rational integral expression in x be divided by $x - n$, the remainder is the same as the original expression with n substituted for x.

Example :

$$\begin{array}{llll}
x^2 - 5x & + 6 & \multicolumn{1}{|l}{x - n} & \\
\cline{3-3}
\underline{x^2 - nx} & & \multicolumn{1}{|l}{x + (n - 5)} & \\
(n - 5)x + 6 & & & \\
\underline{(n - 5)x} & \underline{- n^2 + 5n} & & \\
 & n^2 - 5n + 6, & \text{Remainder.} &
\end{array}$$

Here the remainder $n^2 - 5n + 6$ is the same as $x^2 - 5x + 6$ when n is substituted for x.

Now if n is a letter or a number such that the remainder $n^2 - 5n + 6$ is zero, the division is exact; and the value of n, if substituted for x, will make $x^2 - 5x + 6$ zero also.

Hence, if by trial we can discover a number n which, when put for x, makes $x^2 - 5x + 6$ zero, $x - n$ will be an exact divisor of $x^2 - 5x + 6$. If 2 is put for x in $x^2 - 5x + 6$, we get $4 - 10 + 6$, or zero. Therefore $x - 2$ is a factor of $x^2 - 5x + 6$.

The last paragraph illustrates the following theorem :

116. Factor Theorem. *If any rational integral expression in* x *becomes zero when any number* n *is put for* x, $x - n$ *is a factor of the expression.*

The Factor Theorem may be used to factor many of the preceding exercises. Moreover, many expressions which, by previous methods, are very difficult to factor, may be readily factored by the aid of this theorem.

Note. By means of the Factor Theorem we are able to solve cubic equations when the roots are integers. The solution of the general cubic equation is one of the famous problems of mathematics, and one which is accompanied by many interesting applications. This problem was first solved by the Italian, Tartaglia, about 1530, but was published by Cardan, to whom Tartaglia explained his solution on the pledge that he would not divulge it. For many years the credit for the discovery was given to Cardan, and to this day it is usually called Cardan's solution.

Example 1. Factor $x^3 + x - 2$.

Solution: If $x - n$ is a factor of $x^3 + x - 2$, then n must be an integral divisor of 2. Now the integral divisors of 2 are $+1$, -1, $+2$, and -2. If 1 be put for x, $x^3 + x - 2 = 1 + 1 - 2 = 0$. Therefore $x - 1$ is a factor of $x^3 + x - 2$. Dividing $x^3 + x - 2$ by $x - 1$, the quotient is $x^2 + x + 2$. None of the integral divisors of 2, when put for x, make $x^2 + x + 2$ zero; hence $x^2 + x + 2$ is prime.

Therefore $x^3 + x - 2 = (x - 1)(x^2 + x + 2)$.

Example 2. Factor $x^3 + 2x^2 - 5x - 6$.

Solution: The integral divisors of 6 are $+1$, -1, $+2$, -2, $+3$, -3, $+6$, and -6. If we put 1 for x, $x^3 + 2x^2 - 5x - 6 = 1 + 2 - 5 - 6 = -8$. If we put -1 for x, $x^3 + 2x^2 - 5x - 6 = -1 + 2 + 5 - 6 = 0$. Therefore $x - (-1)$, or $x + 1$, is a factor. Dividing $x^3 + 2x^2 - 5x - 6$ by $x + 1$, the quotient is $x^2 + x - 6$, which equals $(x + 3)(x - 2)$.

Therefore $x^3 + 2x^2 - 5x - 6 = (x + 1)(x - 2)(x + 3)$.

In the following exercises, when searching for the values of x which will make the given expression zero, only integral divisors of the last term of the expression (arranged according to the descending powers of x) need be tried.

EXERCISES

1. Divide $x^2 + bx + c$ by $x - n$ and find the remainder.

2. Find the remainder in $(x^3 + ax^2 + bx + c) \div (x - n)$.

Find, by the Remainder Theorem, the remainder when:

3. $x^3 - x - 8$ is divided by $x + 2$.

4. $x^4 - x + 6$ is divided by $x + 2$.

Factor:

5. $x^3 - 3x + 2$.

6. $x^3 - 4x + 3$.

7. $x^3 + 2x + 3$.

8. $x^3 - x - 6$.

9. $x^3 - x + 6$.

10. $x^3 - 11x + 6$.

11. $x^3 - 11x - 6$.

12. $x^3 - 14x - 8$.

13. $x^3 - 27x - 10$.

14. $x^3 + 3x^2 + 3x + 2$.

15. $x^3 + 4x^2 + 5x + 2$.

16. $x^3 - 6x^2 + 11x - 6$.

17. $x^4 - 11x^2 + 2x + 12$.

18. $4x^4 - 3x - 1$.

19. $x^3 - 5a^2x + 2a^3$.

20. $x^3 - 7m^2x - 6m^3$.

21. $x^3 - 2nx^2 - 5n^2x + 6n^3$.

117. The sum or difference of two like powers. The type form is

$$a^n \pm b^n.$$

The cases in which $a^n \pm b^n$ is divisible by $a+b$ or $a-b$ can be determined by the Factor Theorem.

Thus in $a^n - b^n$, n being either an odd or an even integer, let $a=b$. Then $a^n - b^n$ becomes $b^n - b^n = 0$. Therefore $a-b$ is a factor of $a^n - b^n$.

In $a^n - b^n$, n being even, let $a = +b$ or $-b$. Then $a^n - b^n$ becomes $b^n - b^n = 0$, since both $(+b)^n$ and $(-b)^n$ are *positive* when n is *even*. Therefore when n is even, both $a-b$ and $a+b$ are exact divisors of $a^n - b^n$.

In $a^n + b^n$, n being even, let a equal either $+b$ or $-b$. Then $a^n + b^n$ becomes $b^n + b^n$, which is not zero. Therefore $a^n + b^n$ is never divisible by $a+b$ or $a-b$ when n is even.

In $a^n + b^n$, n being odd, let $a = -b$. Then $a^n + b^n$ becomes $(-b)^n + b^n = 0$, since $(-b)^n$ is *negative* when n is *odd*. Therefore when n is odd, $a+b$ is a divisor of $a^n + b^n$.

Summing up:

I. $a^n - b^n$ is *always* divisible by $a-b$.

II. $a^n - b^n$, when n is *even*, is always divisible by $a-b$ and $a+b$.

III. $a^n + b^n$ is *never* divisible by $a-b$.

IV. $a^n + b^n$, when n is *odd*, is always divisible by $a+b$.

It is worth noting that $a^n + b^n$ is usually prime when n is a power of 2. (See, however, Exercises 21–27, page 294.)

Thus $a^2 + b^2$, $a^4 + b^4$, $a^8 + b^8$, etc., are prime.

In every other case $a^n + b^n$ is not prime.

Thus
$$a^6 + b^6 = (a^2)^3 + (b^2)^3,$$
$$a^{10} + b^{10} = (a^2)^5 + (b^2)^5,$$
$$a^{12} + b^{12} = (a^4)^3 + (b^4)^3, \text{ etc.}$$

If n is even, $a^n - b^n$ is the difference of two squares. Such binomials should first be separated into two factors as on page 293. See also Example 2, page 298.

EXAMPLES

1. Factor $a^5 - b^5$.

Solution: $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

2. Factor $a^6 - b^6$.

Solution: $a^6 - b^6$ is divisible by $a + b$ and $a - b$. It is better, however, to regard all such binomials with even exponents as the difference of two squares. Thus $a^6 - b^6 = (a^3 - b^3)(a^3 + b^3)$, etc.

3. Factor $a^5 + b^5$.

Solution: $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Note the signs of the second factor in Examples 1 and 3, — all plus in one case, alternately plus and minus in the other.

4. Factor $32x^{15} + y^{15}$.

Solution: $32x^{15} + y^{15} = (2x^3)^5 + (y^3)^5$

$$= (2x^3 + y^3)[(2x^3)^4 - (2x^3)^3(y^3) + (2x^3)^2(y^3)^2 - (2x^3)(y^3)^3 + (y^3)^4]$$

$$= (2x^3 + y^3)(16x^{12} - 8x^9y^3 + 4x^6y^6 - 2x^3y^9 + y^{12}).$$

EXERCISES

Factor:

1. $x^5 + 1$.

2. $x^5 + y^5$.

3. $x^7 - y^7$.

4. $x^7 + 1$.

5. $x^5 - 32$.

6. $3125 - c^5$.

7. $c^7 - 128$.

8. $c^7 - x^{14}$.

9. $x^{10} + y^5$.

10. $1 + c^7d^{14}$.

11. $c^{14} + 128$.

12. $x^{10} + a^{15}$.

13. $x^5 - 32y^{10}$.

14. $128x^7 - 1$.

15. $1024 - 243x^5$.

118. General directions for factoring. The student should not hasten to factor an expression until he has observed it carefully and determined to what type it belongs. The following suggestions will prove helpful:

I. *First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.*

II. *Then determine, by the form of the polynomial factor, with which of the following types it should be classed, and use the methods of factoring applicable to that type.*

1. $ax + ay + bx + by.$
2. $a^2 + 2ab + b^2.$
3. $x^2 + bx + c.$
4. $ax^2 + bx + c.$
5. $a^n \pm b^n.$
6. $\left\{\begin{array}{l} a^2 - b^2. \\ a^2 + 2ab + b^2 - c^2. \\ a^2 + 2ab + b^2 - c^2 - 2cd - d^2. \end{array}\right.$
7. $a^4 + ka^2b^2 + b^4.$

III. *Proceed again as in* II *with each polynomial factor obtained until the original expression has been separated into its prime factors.*

IV. *If the preceding steps fail, try the Factor Theorem.*

REVIEW EXERCISES

Factor:

1. $x^3 - x.$
2. $x^{10} - x^2.$
3. $x^8 - 2x^4 + 1.$
4. $x^5 - 8x^3 + 16x.$
5. $x^4 - 10x^2 + 9.$
6. $x^4 - 13x^2 + 36.$
7. $18a^2x^2 - 24a^2x - 10a^2.$
8. $x^2 - \frac{1}{6}x - \frac{1}{6}.$
9. $16x^4 + 8x^2 - 3.$
10. $a^3 - a + a^2b - b.$
11. $3x^4 - 15x^2 + 12.$
12. $12a - 39ay - 51ay^2.$
13. $x^4 - 3x^3 + 4x^2 - 12x.$
14. $2a^3b + 3a^2b - 8ab - 12b.$
15. $4a^2 - a^4 + 81 + 10a^2x - 36a - 25x^2.$
16. $12cd^3 - 6a^3x - a^6 + 4c^2 + 9d^6 - 9x^2.$
17. $x^6 + 1.$
18. $x^5 - \frac{1}{16}a^4x.$
19. $x^{10} - y^{10}.$
20. $x^8 - x^2.$
21. $x^{12} + y^{12}.$
22. $x^{12} - y^{12}.$
23. $x^{12} - 64.$
24. $x^{10} + xy^9.$
25. $x^{12} - 8.$
26. $x^4 - y^{16}.$
27. $64x^{12} - 4x^4.$
28. $x^{12} + 64y^{12}.$
29. $32x^{10} + y^{10}.$
30. $x^{16} - y^{16}.$
31. $16x^{16} - 1.$
32. $5^{-4x} - 1.$
33. $10 - 10c^{14}d^4.$
34. $2cd - c^2 - d^2.$
35. $x^4 - 9x^2 - x + 3.$
36. $x^4 - 7x^2y^2 + 81y^4.$
37. $4c^4 + 20c^3d - 11c^2d^2.$
38. $a^5 - a^3 + a - 1.$
39. $a^5 - a^4 - a^3 + a.$
40. $5d^2 - 5cd - 10c^2.$

41. $x^3 - 3x^2 + 8x - 12.$

42. $121x^4 - 476x^2y^2 + 100y^4.$

43. $x^4 - x^2 + 12xy - 36y^2.$

44. $y^4 - 18y^2 + 81 - 16x^4 - 24x^2y^3 - 9y^6.$

45. $h^5k^5 - 1024k^5.$

46. $x^3 - 83x^5 + 289x^7.$

47. $x^2 - y^2 - x - y.$

48. $289 - 100a^2 - b^2 - 20ab.$

49. $625a^8 - 169d^4 + 78cd^2 - 9c^2.$

50. $x^{3n} - 125y^{6n}.$

51. $4x^4 - 37x^2 + 9.$

52. $256 - 16k^4 + 8h^2k^2 - h^4.$

53. $x^4 + 4.$

54. $x^{12} - 729.$

55. $a^{4x+8} + 64.$

56. $a^4 + 225a^{-4} - 39.$

57. $x^3 - 6x^2 + 12x - 8.$

58. $a^2 - 9d^2 - 8ab + 6cd - c^2 + 16b^2.$

59. $4h^{-6} - 20h^{-3}k + 25k^2 - 6ab^{-2} - 9a^2 - b^{-4}.$

60. $x^{2a} - 2x^a - 15.$

61. $a^3 + a + b^3 + b.$

62. $x^{2a} - 12x^a + 36.$

63. $25x^{2c} + 50x^c - 39.$

64. $128 - x^{28}.$

65. $3x^2 + 10x - 8.$

66. $a^5 - a^3 - a^2 + 1.$

67. $x^3 + 3x^2 + 9x + 27.$

68. $x^3 - 6x^2 + 12x - 7.$

69. $4x^4 - 25y^6 + 10y^3 - 12x^2 + 8.$

70. $e^{2x} - 2 + e^{-2x}.$

71. $e^{2x} - 5 + 6e^{-2x}.$

72. $3x^{2a} + 5x^a - 28.$

73. $e^{3x} - 2e^x - 24e^{-x}.$

74. $6e^{2x} - 5e^{-2x} - 13.$

75. $x^3 - 7xy^2 + 6y^3.$

76. $e^{3x} + e^{2x} + e^x + 1.$

77. $a^{4x-2} - 10 + 25a^{2-4x}.$

78. $e^{3x} - e^{-3x} + 3e^{-x} - 3e^x.$

79. $e^{x+3} + e^{x+2} - e^{3-x} - e^{2-x}.$

80. $xy^2 + xz^2 + x^2y + x^2z + yz^2 + y^2z + 2xyz.$

81. $ab^3 - a^3b + ac^3 - a^3c + bc^3 - b^3c.$

119. Solution of equations by factoring. Study pages 123–124. It is of particular importance that the principles on these pages be thoroughly mastered, and the significance of each step in the solution of the following equations completely grasped.

EXERCISES AND PROBLEMS

Solve by factoring and check:

1. $x^2 - 25 = 0$.
2. $x^2 + 10 = 7x$.
3. $y^2 - 9y = 0$.
4. $r^2 - ra = 30a^2$.
5. $4z^3 - 36z = 0$.
6. $x^4 + 4 = 5x^2$.
7. $t^6 = 13t^4 - 36t^2$.
8. $3x^2 - xb - 2b^2 = 0$.
9. $x^3 - 2x^2 = x - 2$.
10. $x^{2x} - 2x^x + 1 = 0$.
11. $x^{2x} - 8x^x + 16 = 0$.
12. $x^3 + 5x^2 - 18x - 72 = 0$.

Hint. Apply the Factor Theorem.

13. $x^4 + 3x^3 - 8x^2 + 16 = 12x$.
14. $x^3 + 6a^3 = 2ax^2 + 5a^2x$.
15. $x^5 + 9ax^2 = 9x^4a + x^3$.
16. $x^3 + 5x^2c - 16xc^2 - 80c^3 = 0$.

17. Separate 272 into two parts such that the greater equals the square of the less.

18. Twice the square of the least of three consecutive even numbers is 104 greater than the product of the others. Find each.

19. Find a number which when added to four times its reciprocal gives 20.2.

20. A and B together can do a piece of work in $1\frac{7}{8}$ days. If A requires 2 days more than B, how many days does each require alone?

21. The hypotenuse of a right triangle is 40 feet. One leg is 8 feet longer than the other. Find the three sides.

22. The rates of two trains differ by 10 miles per hour. The slower requires 2 hours more than the faster to run 240 miles. Find the rate of each train.

23. The bases of a trapezoid are respectively 2 feet and 8 feet longer than the altitude, and the area is 14 square yards. Find the bases and the altitude.

EXERCISES

Find the H. C. F. of the following:

1. 28, 56, 84, 35.

2. 225, 120, 210, 135.

3. 198, 495, 693, 1155.

4. 816, 1224, 1360, 4080.

5. $91\,x^4y^3$, $133\,x^2y^6$, $343\,x^5y^2$.

6. $a^2 - 9\,a + 14$, $a^2 - 4$, $5\,a^2 - 10\,a$.

7. $x^3 + 27$, $2\,x^2 + 3\,x - 9$, $5\,x^3 + 15\,x^2$.

8. $2\,x^5 + 8\,x$, $3\,ax^3 + 6\,ax + 6\,ax^2$, $3\,ax^7 + 12\,ax^3$.

9. $[(x+y)(x-y)]^3$, $x^4 - 2\,x^2y^2 + y^4$, $(x^6 - y^6)^2$.

10. $x^7 + a^{7e}$, $x^2 - a^{2e}$, $x^2 - 3\,xa^e - 4\,a^{2e}$, $x^3 + a^{3e}$.

11. $a^{5e} + 4\,a^eb^8$, $a^{2e+1}b^2 + 2\,ab^6 - 2\,a^{e+1}b^4$, $a^{8e} - 16\,b^{16}$.

Find the L. C. M. of:

12. 24, 30, 54.

13. 105, 140, 245.

14. 174, 485, 4611, 5141.

15. $30\,ax^2$, $225\,a^5xy^2$, $75\,a^4x^3y$.

16. $12\,x^2 + 6\,x$, $12\,x^3 - 3\,x$, $16\,x^4 + 2\,x$.

17. $a^3 - 8\,b^3$, $4\,b^2 - a^2$, $a^3b + 4\,ab^3 + 2\,a^2b^2$.

18. $x^3 - 2\,x^2 - 5\,x + 6$, $4 - x^2$, $a - ax^2$.

19. $a^4 + 4\,a^2 + 16$, $a^2 - 4$, $a^3 + 8$, $a^3 - 8$.

20. $x^3 - 2\,a^2x + ax^2$, $2\,a^3 + 3\,a^2x + ax^2$, $4\,a^4x^2 - a^2x^4$.

21. $m^4 - 3\,m^2n^2 + 9\,n^4$, $m^3 + 3\,mn^2 + 3\,m^2n$, $3\,n^3 + nm^2 - 3\,mn^2$.

22. $e^{2x} + e^{-2x} - 2$, $e^{2x} - e^{-2x}$, $e^{2x} - 3 + 2\,e^{-2x}$.

23. $x^3 - 2\,x^2 - 2\,x - 3$, $x^3 - 27$, $x^2 + x - 12$.

120. Addition and subtraction of fractions. The student should make whatever review is necessary of the definitions, principles, examples, and rules on pages 135–143.

EXERCISES

Reduce to lowest terms:

1. $\dfrac{18\,a^3c^2}{24\,a^2c^3}$. **3.** $\dfrac{7\,a^2-14\,b^2}{a^4+a^2b^2-6\,b^4}$. **5.** $\dfrac{x^{2a}-c^{2b}}{(x^a+c^b)^2}$.

2. $\dfrac{3\,x-6}{x^2-4}$. **4.** $\dfrac{x^3-8}{(x-2)^3}$. **6.** $\dfrac{(x^2-c^2)^2}{x^4-c^4}$.

7. $\dfrac{2\,x^{6e}-128}{x^{2e}-4}$. **9.** $\dfrac{3\,x^2+2\,x-21}{27\,x^4-147\,x^2}$.

8. $\dfrac{x^6-1}{x^4+x^2+1}$. **10.** $\dfrac{x^4-x^2+x-1}{x^4-1}$.

By the use of § 62 write in three other ways:

11. $-\dfrac{c}{-2\,a}$. **12.** $\dfrac{c}{a-2\,c}$. **13.** $\dfrac{2\,x-3\,y}{3\,x^2-6\,y^2-xy}$.

Change to respectively equivalent fractions, writing the letters in the denominators in alphabetical order and making the first term in each factor positive:

14. $\dfrac{-3}{(c-a)(b-a)}$. **15.** $\dfrac{x-y}{(y-x)(z-x)(z-y)}$.

16. Does $\dfrac{x-3}{5-x-x^2}=\dfrac{x+3}{x^2+x-5}$? Why?

Change to respectively equivalent fractions having the lowest common denominator:

17. $\frac{7}{24}$, $\frac{9}{56}$. **18.** $\dfrac{4}{ab}$, $\dfrac{3}{a^2c}$. **19.** $\dfrac{3\,a+b}{6\,a}$, $\dfrac{a-2\,b}{4\,ab}$.

20. $\dfrac{2}{6\,x-12}$, $\dfrac{3}{x^2-2\,x}$. **21.** $\dfrac{x-1}{x^2-5\,x+6}$, $\dfrac{x}{x^2-9}$.

22. Does $\dfrac{2\,x+5}{3\,a+5}$ equal $\dfrac{2\,x}{3\,a}$? Explain. Define cancellation.

23. If the same term occurs in each member of an equation, may it be canceled? Explain.

Find the algebraic sum of:

24. $\frac{3a}{6}-\frac{a-x}{9}-\frac{3x}{4}$.

26. $\frac{5-x}{x-4}-\frac{5}{7}$.

25. $\frac{x-3}{4x^2}-\frac{3x^2-2}{14x^3}-\frac{x^3-8}{3x^4}$.

27. $\frac{4m^2}{x^2-9m^2}-\frac{3m-x}{x+3m}$.

28. $\frac{x-3}{x^2-9x+14}-\frac{3}{x^2-4}$.

29. $\frac{x}{x^2-2x}-\frac{1}{x}-\frac{2+x}{x^2-4x+4}$.

30. $2-\frac{1}{v-1}+\frac{1}{1-v}$.

31. $x^2+x+1-\frac{x^3}{x-1}$.

32. $x^2-\frac{x^4-2x^2}{x^2-x+1}+x+1$.

33. $\frac{2x^2-3a^2}{27x^3-64a^3}-\frac{2x+a}{9x^2-16a^2}$.

34. $\frac{2}{(a-c)(x-c)}-\frac{3}{(c-a)(c-x)}$.

35. $\frac{2x-4}{9-x^2}-\frac{3x+1}{x-3}-\frac{x}{9-6x+x^2}$.

36. $\frac{2x+12c}{6x^2-13cx-5c^2}-\frac{3x-7c}{4x^2+4cx-35c^2}$.

37. $\frac{(a-s)(c-s)}{s(s-m)}+\frac{ac}{ms}+\frac{(a-m)(c-m)}{m(m-s)}$.

38. $\frac{2x^3-x^2}{x^2-x+1}-\frac{x^2-3}{x^4+x^2+1}+\frac{3x-5}{x^3+1}$.

Show that:

39. $\frac{e^{3x}-e^{-3x}}{e^x-e^{-x}}=5\frac{1}{4}$, when $e^{2x}=4$.

40. $c(c-y)\div d-(y+d)d\div c=y$, when $y=c-d$.

41. $\frac{a}{b}-\frac{c}{d}=\frac{16}{x^4+4x^2+16}$, when $a=x+2$, $b=x^2+2x+4$, $c=x-2$, and $d=x^2-2x+4$.

121. Multiplication and division of fractions. The student should make any necessary review of pages 147–154.

The **reciprocal** of a number is 1 divided by the number.

Thus the reciprocal of 2 is $\frac{1}{2}$; of $\frac{3}{5}$ is $\frac{5}{3}$; and of $3\frac{2}{7}$ is $\frac{7}{23}$.

Therefore the quotient of one fraction by another is the product of the first and the reciprocal of the second. Also the quotient of a fraction by an integral expression is the product of the fraction and the reciprocal of the integral expression.

EXERCISES

Perform the indicated operations:

1. $\frac{4x^2c^2}{9a^3}\cdot\frac{27a^2}{10x^3c}$. 3. $\left(\frac{a}{6c^2}\right)\cdot\left(\frac{3a}{c}\right)^2\cdot\left(\frac{2c^2}{3a^4}\right)^2$.

2. $\left(\frac{2a}{cx}\right)^2\cdot\left(\frac{3c^2x}{4a^2}\right)^3$. 4. $\left(\frac{-a}{x}\right)^3\cdot\left(\frac{-2x}{a}\right)^2\cdot\left(\frac{-c}{2a}\right)^4$.

5. $\left(\frac{3c^2}{2a}\right)^2\div\left(\frac{9c^2x^3}{4a^2}\right)^2\div\left(\frac{-2a^3c^2}{5x^3}\right)^3\frac{(3a^2c)^4}{(10x^3)^2}$.

6. $\frac{4x^e-2y^e}{(2x^e-y^e)^2}\cdot\frac{4x^{2e}-y^{2e}}{(2x^e+y^e)^2}\cdot\frac{2x^e+y^e}{3}$.

7. $\frac{x^2-1}{x^4-1}\left(1-\frac{2}{x^6+1}\right)\frac{x^4-x^2+1}{x^2+x+1}$.

8. $\left(x^2-2x+4-\frac{16}{x+2}\right)\frac{x^2-4}{3}$.

9. $\frac{a^2-a-90}{a^2-100}\div\frac{a^3+9a^2}{a^2+10a}\div\frac{4a+6}{2a^2x+3ax}$.

10. $\left(a-\frac{13a}{a+6}\right)\div\frac{a^2-a-42}{a^2-36}\div\frac{2a^3-12a}{ax+6x}$.

11. $\left(\frac{3a^4-75a^2}{3a-7}\right)\left(6a-\frac{7}{a}-11\right)\div\left(2+\frac{5}{a^2}+\frac{11}{a}\right)$.

12. $\left(\frac{4c}{a}-\frac{15c^2}{a^2}+4\right)\left(3-\frac{4a+20c}{2a+5c}\right)\div\left(4-\frac{16c}{a}+\frac{15c^2}{a^2}\right)$.

13. $\frac{1}{mx}\left(\frac{m}{x}+\frac{x}{m}\right)\div\left(\frac{m^6+x^6}{m^3x^3}\right)\left(m^2-x^2+\frac{x^4}{m^2}\right)\left(\frac{am+mx}{cx-ax}\right)$.

14. Multiply $\frac{x^3-8}{x^4+4x^2+16}\cdot\frac{x^2-4}{(x-2)^2}$ by the reciprocal of $\frac{(x+2)^2}{x^3+8}$.

15. $$\frac{2x^2+5x+3}{6x^2+x-1}\cdot\frac{3x^2+3ax-x-a}{2x^2-2cx+3x-3c}\cdot\left(\frac{c+2a+(2a+1+2c)x}{x^2+x+ax+a}-2\right).$$

Simplify each term in:

16. $$\left(\frac{2x^3}{c^2}\right)^5-5\left(\frac{2x^3}{c^2}\right)^4\left(\frac{c^3}{15x^4}\right)+10\left(\frac{2x^3}{c^2}\right)^3\left(\frac{c^3}{15x^4}\right)^2-10\left(\frac{2x^3}{c^2}\right)^2\left(\frac{c^3}{15x^4}\right)^3.$$

17. $$\left(\frac{a^2}{b^3}\right)^{10}-10\left(\frac{a^2}{b^3}\right)^9\left(\frac{b^2}{6a^4}\right)+\frac{10\cdot 9}{2}\left(\frac{a^2}{b^3}\right)^8\left(\frac{b^2}{6a^4}\right)^2-\frac{10\cdot 9\cdot 8}{2\cdot 3}\left(\frac{a^2}{b^3}\right)^7\left(\frac{b^2}{6a^4}\right)^3.$$

18. Explain why the divisor is inverted in division of fractions.

19. Show how the rule for the division of fractions is based on the first principle of this chapter.

Simplify:

20. $\dfrac{8-\frac{1}{27}}{2-\frac{1}{3}}$.

21. $\dfrac{7\frac{3}{7}}{40\frac{5}{9}}\cdot\dfrac{146}{17\frac{1}{3}}$.

22. $\dfrac{3\frac{1}{2}\cdot 3\frac{1}{2}\cdot 3\frac{1}{2}-1}{3\frac{1}{2}\cdot 3\frac{1}{2}-1}$.

23. $\dfrac{\frac{a}{x}+3}{\frac{a^2}{x^2}-9}$.

24. $\dfrac{1-\frac{m}{x}}{\frac{4m^4}{x^4}-4}$.

25. $\dfrac{2a-\frac{a^2+c^2}{c}}{\frac{a}{c}-1}$.

26. $\dfrac{3-\frac{a}{c}}{\frac{(a+3c)^2}{2ac}-6}$.

27. $\dfrac{c-\frac{4}{c}}{\frac{1}{c^2}+\frac{2}{c^3}-\frac{8}{c^4}}$.

28. $\dfrac{\left(\frac{4c+3a}{3a}\right)^2-\frac{6c}{a}}{\frac{(3a+2c)^2}{3a}-8c}$.

29. $1+\dfrac{1}{1+\frac{1}{1+2}}$.

30. $1-\dfrac{2}{3+\dfrac{4}{5-\frac{6}{7}}}$.

31. $\dfrac{\dfrac{m^2}{n^2}+\dfrac{mn}{mn}+\dfrac{n^2}{m^2}}{\dfrac{m^2+n^2}{mn}-1}$.

32. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$.

33. $3-\dfrac{1}{1-\dfrac{4}{6+\dfrac{3}{7-\frac{2}{5}}}}$.

34. $\dfrac{6x-11-\dfrac{7}{x}}{2+\dfrac{11}{x}+\dfrac{5}{x^2}}\div\dfrac{\dfrac{5-x}{3x}}{x^2-25}$.

35. $\dfrac{1-\dfrac{a^2b^2}{(a^2+b^2)^2}}{\dfrac{ab}{a^2+b^2}+1}$.

36. $\dfrac{1+\dfrac{y}{x}+\dfrac{x}{y}}{\dfrac{x^2}{y^2}+1+\dfrac{y^2}{x^2}}$.

37. $\dfrac{\dfrac{1}{x}-\dfrac{x-a}{x^2-a^2}}{\dfrac{1}{a}-\dfrac{a-x}{a^2+x^2}}$.

38. $\dfrac{\dfrac{xy+1}{y}}{x+\dfrac{1}{\dfrac{yz+1}{z}}}-\dfrac{1}{y(x+xyz+z)}$.

39. $\dfrac{1}{1+\dfrac{a}{a+1+\dfrac{2a^2}{1-a}}}$.

40. What value has $\dfrac{a^3-b^3}{a^2+b^2}$ when $a=e^x-\dfrac{1}{e^x}$ and $b=e^x+\dfrac{1}{e^x}$?

41. Brouncker (1620–1684) proved that π (the circumference of a circle divided by its diameter) is four times the fraction on the right.

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2}}}}}}+\text{etc.}$$

(*a*) Rewrite the fraction, continuing it to $\frac{225}{2}$ etc.

(*b*) Stopping with $2+\frac{81}{2}$, find the difference between four times this fraction and the value of π, 3.1416.

Note. William Brouncker, one of the brilliant mathematicians of his time, was an intimate friend of John Wallis (see page 356). These two scientists were among the pioneers in the study of expressions with an infinite number of terms.

The complex fraction in the exercise, if continued indefinitely according to the law which its form suggests, is called an infinite continued fraction. Brouncker was the first to study such expressions.

CHAPTER XXVI

LINEAR EQUATIONS IN ONE UNKNOWN

(In Part Review)

122. Use of the axioms in solving equations. Two or more equations, even if of very different form, are **equivalent** if all are satisfied by every value of the unknown which satisfies any one.

Of the four axioms, or assumptions (see page 33), we shall make constant use. If the "same number" referred to in each is expressed arithmetically, the result is always an equation *equivalent* to the *original* one. Further, if *identical expressions involving the unknown* be added to or subtracted from each member of an equation, the resulting equation is equivalent to the first. If, however, both members of an equation be multiplied by or divided by identical expressions containing the unknown, the resulting equation *may not* be equivalent to the original one. In other words, under the condition just stated, roots may be introduced or lost by the use of Axiom III or IV respectively. The examples which follow illustrate the important facts concerning the introduction or loss of roots which enter or disappear even though a proper use of the axioms is made.

Example 1. Let $x - 2 = 7.$ (1)

Multiplying by $x - 3$, $x^2 - 5x + 6 = 7x - 21.$ (2)

From (2), $x^2 - 12x + 27 = 0.$ (3)

Whence $(x - 3)(x - 9) = 0.$ (4)

Therefore $x = 3 \text{ or } 9.$

Since (1) has the root 9 only, and (3) has the two roots 3 and 9, (3) is not equivalent to (1), that is, a root was introduced by the use of the multiplication axiom.

Example 2. Let $x^2 - 4 = x + 2.$ (1)
Dividing by $x + 2$, $x - 2 = 1.$ (2)
Solving (2), $x = 3.$ (3)
But from (1), $x^2 - x - 6 = 0.$ (4)
Whence $(x - 3)(x + 2) = 0.$ (5)
Therefore $x = 3$ or $-2.$

Here (5) shows that (1) has the two roots 3 and -2, and since (2) has but one root, 3, it is evident that a root was lost by the use of the division axiom.

The student should note the preceding illustration carefully, as the possibility of dividing each member of an equation by a common factor involving the unknown frequently arises. A very common type is the following:

Example 3. Let $x^2 - 2x = 0.$ (1)
Dividing by x, $x - 2 = 0.$ (2)
Whence $x = 2.$ (3)

But (1) has the root 0 also, which is lost by dividing both members of (1) by x.

If proper methods of solution are applied to an equation (or to a false statement in the form of an equation), and one or more values of the unknown which are thus obtained do not satisfy the original statement, such values are called **extraneous** (or extra) roots.

An extraneous root is a root of an equation which is not equivalent to the original statement, but of one which is derived from the original statement in the process of solution.

123. Principles. The preceding discussion may be summed up thus:

PRINCIPLE I. *If identical expressions (which may or may not contain the unknown) be added to or subtracted from each member of an equation, the resulting equation is equivalent to the original one.*

PRINCIPLE II. *Extraneous roots may be introduced into a solution by multiplying both members of an equation by an integral expression containing the unknown.*

Principle III. *If both members of an equation be divided by an integral expression containing the unknown, one or more roots will usually be lost by such a division.*

It can be seen from Example 1 that the root introduced is the value of the unknown obtained by setting the multiplier, $x - 3$, equal to zero and solving the resulting equation.

Similarly, it can be seen from Example 2 that the root lost is that value of the unknown obtained by setting the divisor, $x + 2$, equal to zero and solving the resulting equation.

Sometimes a statement in the form of an equation has no root; yet the ordinary method of solution appears to give one. For example, consider the statement $\frac{4x-1}{x-3} = \frac{x+8}{x-3} + 5$.

If one multiplies each member by $x - 3$ and solves as usual, he obtains $x = 3$. This answer cannot be verified because $x - 3$, the denominator, becomes zero for $x = 3$. Here the multiplication by $x - 3$ introduced the value 3 for x. Checking will always discover the falsity of such a result (see page 161). Extra roots usually occur in the solution of equations (or in attempts to solve false statements in equation form) which involve fractions or radicals.

For solving equations in one unknown which may or may not involve fractions we have the

Rule. *Free the equation of any parentheses it may contain except such as inclose factors of the denominators.*

Where polynomial denominators occur, factor them if possible.

Find the lowest common multiple of the denominators of the fractions and multiply each fraction and each integral term of the equation by it, using cancellation wherever possible.

Transpose, solve as usual, and reject all values for the unknown which do not satisfy the original equation.

Checking the solution of an equation is often called **testing,** or **verifying,** the result. For this we have the

Rule. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify the result until the two members are seen to be identical.*

EXERCISES

1. Define and illustrate: equation, identity, equation of condition, linear equation in one unknown, satisfy, root, extraneous root, substitute, verify, check, and axiom.

2. Give an example of (*a*) a numerical identity; (*b*) a literal identity; (*c*) a conditional equation; (*d*) two equivalent equations; (*e*) a statement in the form of an equation, but which has no root.

3. Define and illustrate transposition. (*a*) On what axiom does transposition depend? (*b*) If one equation is obtained from another by transposition, are the two equivalent? Explain.

4. What extraneous roots, if any, are introduced if both members of the following equations are multiplied by the expression on the right?

(*a*) $x + 3 = 7$	$x + 4$
(*b*) $x + 5 = 0$	$x + 5$
(*c*) $x + c = 0$	$x - c$
(*d*) $x + a = 0$	x
(*e*) $x = 5$	4
(*f*) $x - 1 = 0$	$x^2 + 3x + 2$

5. What roots, if any, are lost by dividing both members of the following equations by the expression on the right?

(*a*) $x^2 - 4 = 0$	$x - 2$
(*b*) $x^2 - 4x + 3 = x - 3$	$x - 3$
(*c*) $x^2 + x - 12 = x + 4$	$x + 4$
(*d*) $x^2 - 2x = 4x$	x
(*e*) $x^4 - 16 = x^2 - 4$	$x^2 - 4$
(*f*) $(x - a)^3 = (x - a)^2$	$x - a$

Solve the following for the unknowns involved, considering *a*, *b*, *c*, and *d* as known numbers:

6. $\frac{5x}{21} - \frac{a}{6} = \frac{x}{14}$. **7.** $\frac{3}{2x} - \frac{3}{20} = \frac{7}{5x}$. **8.** $\frac{m-2}{m-3} = \frac{17}{18}$.

9. $\dfrac{2x+5}{10x} - \dfrac{3(2x+1)}{2x} = 3\tfrac{1}{5}.$

10. $\dfrac{5x-7}{6} - \dfrac{5}{2}\left(\dfrac{4-x}{10}\right) = \dfrac{15x-22}{6}.$

11. $\dfrac{1-8x}{5} - \dfrac{2(1-6x)}{24x-3} = \dfrac{1-24x}{15}.$

12. $\dfrac{3x-9}{3x-5} - 2 = \dfrac{3x-5}{8-3x}.$

13. $\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{1}{c} = \dfrac{1}{x}.$

14. $\dfrac{x}{a} + \dfrac{x+m}{b} = 1.$

15. $\dfrac{cy}{2d} - c^2 = \dfrac{2dy - c^3}{c}.$

16. $\dfrac{2x-3b}{a} - \dfrac{2a-3x}{b} + \dfrac{9b}{a} = \dfrac{4a}{b} - 5.$

17. $\dfrac{3}{b(a-x)} + \dfrac{1}{a-x} = \dfrac{-3(b+3)}{2ab}.$

18. $\dfrac{3}{2y+1} - \dfrac{1+2y}{2y-1} = \dfrac{4y^2}{1-4y^2}.$

19. $\left(1 - \dfrac{1}{c}\right) \div \left(1 + \dfrac{1}{z}\right) + \dfrac{c+z}{z+1} = 1.$

20. $\dfrac{3b+9x}{9a+6x} - \dfrac{a-2x}{2x+3a} = 2.$

21. $\dfrac{8}{x-7} - \dfrac{2-6x}{x^2-6x-7} = \dfrac{27}{x+1}.$

22. $\dfrac{x}{2(a+b)} - \dfrac{5}{b-a} = \dfrac{bx}{b^2-a^2}.$

23. $\dfrac{2x}{3} - \dfrac{d}{5}\left(\dfrac{6x}{c} - 10d\right) = 2cd\left(\dfrac{2}{3} - \dfrac{d}{5c}\right).$

24. $82.4 - 13x = 32x - 52.6.$

25. $.01(2x + .205) - .0125(1.5x - .5) = .01955.$

26. $\dfrac{1}{x - .33a} + \dfrac{1}{\dfrac{2x}{3} - .22a} = \dfrac{5}{4a}.$

27. $\dfrac{c^2}{ax} + \dfrac{1}{c} = \dfrac{3c-3a}{x} + \dfrac{1}{a} + \dfrac{a^2}{cx}.$

PROBLEMS

1. At what time between 4 and 5 o'clock will the hands of a clock be together?

Solution: First, the minute hand moves twelve times as fast as the hour hand. Second, at 4 o'clock the hour hand is 20 minute spaces ahead of the minute hand. Now let x equal the number of minute spaces that the minute hand travels from its position at 4 o'clock until it overtakes the hour hand. Obviously the hour hand must travel $x - 20$ spaces before it is overtaken by the minute hand.

Therefore $x = (x - 20)12.$

Whence $x = 21\tfrac{9}{11}.$

Hence the hands are together at $21\tfrac{9}{11}$ minutes after 4 o'clock.

2. At what time between 7 and 8 o'clock are the hands of a clock together?

3. At what time between 2 and 3 o'clock are the hands of a clock in a straight line?

4. At what time between 6 and 7 o'clock is the minute hand (*a*) 10 minute spaces ahead of the hour hand? (*b*) 10 minute spaces behind the hour hand?

5. At what times between 5 and 6 o'clock are the hands of a clock at right angles?

6. If the earth is between a planet and the sun and in a line with them, the planet is said to be in *opposition.* The earth and Mars revolve about the sun in (approximately) 365 days and 687 days respectively. Mars was in opposition September 24, 1909. What is the approximate date of the next opposition?

Solution: For the sake of simplicity we will suppose in this and in similar problems that the planets move in the same plane and in circular paths, of which the sun is the center.

Let $x =$ the required number of days.

Now in x days the earth will make $\frac{x}{365}$ revolutions about the sun.

And in x days Mars will make $\frac{x}{687}$ revolutions about the sun.

But to be in opposition the earth must in x days go round the sun once more than Mars does.

Therefore $$\frac{x}{365} = \frac{x}{687} + 1.$$

Clearing, $$687\,x = 365\,x + 250755.$$

Whence $$x = 779 + .$$

Therefore the required date is November 11, 1911.

7. If a planet is between the earth and the sun and in a line with them, it is said to be in *conjunction.* Venus was in (superior) conjunction April 28, 1909. If Venus revolves about the sun once in 225 days, find the approximate date of the next conjunction.

8. Jupiter revolves about the sun once in 4332 days. On February 28, 1909, the planet was in opposition. Find the approximate date of the next opposition.

9. Two men travel in the same direction around an island, one making the circuit every $2\frac{1}{2}$ hours and the other every 3 hours. If they start together, after how many hours will they be together again?

10. Three automobiles travel in the same direction around a circular road. They make the circuit in $2\frac{3}{4}$ hours, $3\frac{1}{3}$ hours, and $4\frac{2}{5}$ hours respectively. If they start at the same time, after how many hours are the three together again?

11. Is the answer to Exercise 9 an integral multiple of $2\frac{1}{2}$ and 3? Is it the least integral multiple?

12. Is the answer to Exercise 10 an integral multiple of $2\frac{3}{4}$, $3\frac{1}{3}$, and $4\frac{2}{5}$? Is it the least integral multiple?

13. Reduce $2\frac{3}{4}$, $3\frac{1}{3}$, and $4\frac{2}{5}$ to improper fractions and divide the L.C.M. of the numerators by the G.C.D. of the denominators. Compare the result with the answer to Exercise 10.

14. The method of finding the L.C.M. of two or more fractions or mixed numbers is hinted at in Exercise 13. State a rule therefor. Find by the rule the L.C.M. of $1\frac{1}{6}$, $2\frac{1}{3}$, and $3\frac{1}{9}$.

15. Find by the same rule the L.C.M. (*a*) of $\frac{a}{b}$ and $\frac{c}{d}$; (*b*) of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

16. How many ounces of alloy must be added to 56 ounces of silver to make a composition 70% silver?

17. Gun metal of a certain grade is composed of 16% tin and 84% copper. How much tin must be added to 410 pounds of this gun metal to make a composition 18% tin?

HINT. Since the composition is 16% tin, then $\frac{16}{100}\cdot 410$ = the number of pounds of tin in the first composition.

Let x = the number of pounds of tin to be added.

Then $\dfrac{16\cdot 410}{100} + x$ = the number of pounds of tin in the second composition, and $410 + x$ = the number of pounds of both metals in the second composition.

Therefore $$\frac{\dfrac{16\cdot 410}{100} + x}{410 + x} = \frac{18}{100}, \text{ etc.}$$

18. A 30-gallon mixture of milk and water tests 16% cream. How many gallons of water has been added if the milk is known to test 20% cream?

19. How many gallons of alcohol 90% pure must be mixed with 10 gallons of alcohol 95% pure so as to make a mixture 92% pure?

20. The diameter of the earth is $3\frac{2}{3}$ times that of the moon, and the difference of the two diameters is 5760 miles. Find each diameter in miles.

21. The diameter of the sun is 3220 miles greater than 109 times the diameter of the earth, and the sum of the two diameters is 874,420 miles. Find each diameter in miles.

22. The distance of the earth from the sun is $387\frac{1}{2}$ times the earth's distance from the moon. Light traveling 186,000 miles per second would require 8 minutes $18\frac{22}{31}$ seconds longer to go from the earth to the sun than from the earth to the moon. Find each distance in miles.

23. The diameter of Jupiter is $10\frac{10}{11}$ times the diameter of the earth, and the sum of their diameters is 94,320 miles. Find each diameter in miles.

24. A can do a piece of work in 15 days and B in 25 days. After they have worked together 3 days, how many days will B require to finish the work?

25. A can do a piece of work in a days, B in b days, and C in $a + b$ days. How many days will it take them all working together to do the work?

26. A cistern has two pipes. By one it can be filled in $2m$ hours; by the other it can be emptied in $\frac{n+1}{3}$ hours. Assume $2m$ less than $\frac{n+1}{3}$ and find the number of hours required to fill the cistern if both pipes are opened.

27. Discuss Problem 26 thus: What is the relation between m and n if (a) the water could run out more slowly than it comes in; (b) the water could run out as fast as it comes in; (c) the water could run out faster than it comes in?

28. If both pipes in Problem 26 had been intake pipes, how many hours would have been required to fill the cistern one-xth full?

29. If the radius of a circle is increased 7 inches, the area is increased 440 square inches. Find the radius of the first circle ($\pi = \frac{22}{7}$ approximately).

Facts from Geometry. The area of a circle is the square of the radius multiplied by π ($\pi = 3.1416$ approximately). This is expressed by the formula $A = \pi R^2$.

The circumference of a circle equals the diameter times π. The usual formula is $C = 2\pi R$.

30. Imagine that a circular hoop one foot longer than the circumference of the earth is placed about the earth so that it is everywhere equidistant from the equator and lies in its plane. How far from the equator will the hoop be?

31. Compare the result of Exercise 30 with the one obtained when a similar process is carried out on a flagpole 6 inches in diameter, instead of the earth.

32. A passenger train whose rate is 42 miles per hour leaves a certain station a hours and b minutes after a freight train. The passenger train overtakes the freight in b hours and a minutes. Find the rate of the freight train in miles per hour.

33. The arms of a lever are 3 feet and 4 feet in length respectively. What weight on the shorter arm will balance 100 pounds on the longer?

34. A beam 12 feet long supported at each end carries a load of 3 tons at a point 5 feet from one end. Find the load in tons (excluding the weight of the beam itself) on each support.

35. The arms of a balanced lever are 8 feet and 12 feet respectively, the shorter arm carrying a load of 24 pounds. If the load on the longer arm be reduced 4 pounds, how many feet from the fulcrum must an 8-pound weight be placed on the longer arm to restore the balance?

36. A horizontal beam 12 feet long of uniform cross section is hinged at one end and rests on a support which is 4 feet from the other. The free end carries a load of 130 pounds. Excluding the weight of the beam itself, what is the weight in pounds on the support?

HINT. The products of the upward and downward pressures by their respective arms are equal.

37. A 14-foot horizontal beam of uniform cross section weighing 200 pounds is hinged at one end and rests on a support at the other end. (a) What is the weight in pounds on the support? (b) If the support is moved in 3 feet from the end of the beam, find the pressure in pounds on the support.

38. A 16-foot horizontal beam of uniform cross section weighs 300 pounds. It rests on two supports, one at one end and the other 4 feet from the other end. Find the weight in pounds on each support.

CHAPTER XXVII

DETERMINANTS AND REVIEW OF LINEAR SYSTEMS

124. Graphical solution of linear systems. The graph of a linear equation in two variables is a straight line. Therefore it is necessary in constructing the graph of such an equation to locate only two points whose coördinates satisfy the equation and then to draw through the two points a straight line. It is usually most convenient to locate the two points where the line cuts the axes. If these two points are very close together, however, the direction of the line will not be accurately determined. This error can be avoided by selecting two points at a greater distance apart.

The **graphical solution** of a linear system in two variables consists in plotting the two equations to the same scale and on the same axes, and obtaining from the graph the values of x and y at the point of intersection. Two straight lines can intersect in but one point. Hence but one pair of values of x and y satisfies a system of two independent linear equations in two variables.

Through the graphical study of equations we unite the subjects of geometry and algebra, which have hitherto seemed quite separate, and learn to interpret problems of the one in the language of the other.

The student should make such a review of the definitions, illustrations, and theory on pages 187–200 as will enable him to solve the following exercises.

EXERCISES

Solve graphically:

1. $2x + y = 8,$
$x + 2y = 7.$

2. $x - y = 6,$
$3x + 4y = 4.$

3. $x + 2y + 11 = 0,$
$y - x = 2.$

4. $x + 2y = 0,$
$8y + 2x = 3.$

5. $x + 5 = -3y,$
$6y + 2x - 8 = 0.$

6. $2x + 4y = 20,$
$2y - 10 = -x.$

7. $x + y = 5,$
$y + 2 = 0.$

8. $y + 4 = 0,$
$2 - x = 0.$

In Exercise 9 graph each equation. Then add or subtract the corresponding members of the two equations and graph the resulting equation to the same scale on the same axes. Note the position of the third graph with reference to the other two. Proceed in like manner with Exercises 10 and 11.

9. $x + y = 4,$ $x + 2y = 7.$ **10.** $x - y = 5,$ $3x + 2y = 5.$ **11.** $3x - 4y = 12,$ $4x + 3y = -6.$

12. In each of the last three exercises will the values of the x- and y-coördinates of the point of intersection of the two lines, as obtained from the graph, verify in the third equation obtained by adding the two given equations? Why?

13. Graph the equation $x - 2y = 2$. Then multiply both members by 2 or 3 and graph the resulting equation. Compare the two graphs. Then try -2 or -3 as a multiplier and graph the resulting equation. Compare the three graphs. What conclusion seems warranted?

14. What are the coördinates of the origin?

15. Is a graphical solution of a linear system in two variables ever impossible? Explain.

16. In the example on page 198 could different scales have been used on the two axes? Could the two lines have been plotted to different scales? Explain.

17. What is the form of the equation of a line parallel (*a*) to the x-axis? (*b*) to the y-axis?

18. What is the form of the equation of a line through the origin?

19. Give an example of a system in two unknowns which has (*a*) no graphical solution; (*b*) an infinite number of sets of roots.

20. The boiling point of water on a Centigrade thermometer is marked 100°, and on a Fahrenheit 212°. The freezing point on the Centigrade is zero and on the Fahrenheit is 32°. Consequently a degree on one is not equal to a degree on the other, nor does a temperature of 60° Fahrenheit mean 60° Centigrade. Show that the correct relation is expressed by the equation

$C = \frac{5}{9}(F - 32)$, where C represents degrees Centigrade and F degrees Fahrenheit. Construct a graph of this equation. Can you, by means of this graph, express a Centigrade reading in degrees Fahrenheit, and vice versa?

21. By means of the graph drawn in Exercise 20 express the following Centigrade readings in Fahrenheit readings, and vice versa: (*a*) 60°C; (*b*) 150°F; (*c*) – 20°C; (*d*) – 30°F.

22. What reading means the same temperature on both scales?

23. A boy starts at the southwest corner of a field and walks 20 rods, keeping twice as far from the south fence as from the west fence. He then walks east until he is three times as far from the west fence as he is from the south fence. Lastly he walks north until he is as far from one fence as he is from the other. Construct a graph of his path. Find (by measurement) the length of each portion of it and his distance from the starting point.

125. Elimination. The process of deriving from a system of n equations a system of $n - 1$ equations, containing one variable less than the original system, is called **elimination.**

If one equation of a system can be obtained from one or more of the other equations of the system by the direct application of one or more of the axioms, it is called a **derived** equation; if it cannot be so obtained, it is called **independent.**

Only two methods of elimination will be considered, — that of *addition* or *subtraction*, and that of *substitution.*

The student should review the definitions, examples, and rules on pages 203–209.

EXERCISES

1. What is a constant? a variable?

2. Define and give examples involving two unknowns of (*a*) a linear equation; (*b*) a system of linear equations; (*c*) a simultaneous linear system; (*d*) equivalent equations; (*e*) a determinate system; (*f*) an indeterminate equation; (*g*) an indeterminate system; (*h*) an incompatible or inconsistent system.

3. What is the graph in each case in Exercise 2?

4. What is the general form of a linear equation in two variables?

5. To what general form may any incompatible linear system in two unknowns be reduced?

6. What is the general form of a linear system in two unknowns which has an infinite number of sets of roots?

Solve by addition or subtraction:

7. $2x + 5y = 8,$
$x - 10y = 9.$

8. $5x + 38 = 12y,$
$3x + 8y = 0.$

9. $9t - 2n = 18,$
$20t = 7n + 63.$

10. $11m - 10 = -18n,$
$9m + 12n = -15.$

11. $3x - 2y = 18,\ 30 + 8y = 5x.$

Solve by substitution:

12. $3r - 8s = 13,$
$r + 6s = 0.$

13. $2(x + y) + 3y = 4,$
$5 = x + y.$

14. $16x + 7 = 15y,$
$4x + 5y = 0.$

15. $6x + \frac{14}{3} - 12y = 0,$
$7y - 3x - 4 = 0.$

16. $\frac{8m - 3n}{2} + 6n = -9,$
$4m - 1 = 3n.$

Solve by either of the preceding methods:

17. $\frac{2x}{3} + y = -7,$
$5x - 3y = 105.$

18. $\frac{3r}{4} - \frac{7}{2} = \frac{s}{12},$
$r + 8 = -2s.$

19. $\frac{3y + 1}{13} - \frac{z + 20}{12} = -3,$
$\frac{z}{8} - \frac{2y}{9} = 1.$

20. $5x + .7y = -1.25,$
$.12x - .08y = 3\frac{3}{5}.$

21. $\frac{1}{x} + \frac{1}{y} = -\frac{1}{6},$
$\frac{2}{x} - \frac{3}{y} = -\frac{4}{3}.$

22. $\frac{2m + 3n - 2}{m + n + 6} = \frac{4}{3},$
$\frac{1}{m} + \frac{1}{n} = \frac{9}{m}.$

23. $\frac{.25h + 8 + .1k}{k - 10 + h} = 2.5,$
$\frac{1}{.8h - 2.2} + \frac{20}{35 - 5.5k} = 0.$

24. $$\tfrac{2}{3}(2h+5k)=39,\quad 2h-\frac{3k}{4}=3(k-h).$$

25. $$\frac{3(x+y)}{\frac{3}{5}}+\frac{x-y}{-1\frac{1}{5}}=0,\quad 2x+y=7.$$

26. $$\frac{m+5}{n}=\frac{m-5}{n-10},\quad \frac{m-\frac{2}{5}(3m-2n)-\frac{1}{3}}{m-2n-1}+\frac{1}{3}=0.$$

Solve for x and y:

27. $$2cx-y=5ac,\quad \frac{2x}{3}-\frac{y}{c}=a.$$

28. $$\frac{a}{x}+\frac{3a}{y}=1,\quad \frac{3a}{x}+\frac{a}{y}=\frac{1}{2}.$$

29. $$\frac{x+y}{10a}=2-\frac{x-y}{4a},\quad x=y.$$

30. $$ay+(a+b)x=xy,\quad \frac{a+b}{x}+\frac{b}{y}=\frac{4}{3}.$$

31. $$(c+b)y+ax=1,\quad ay=1-(c+b)x.$$

32. $$a^4\cdot a^{y-7}=a^{12},\quad c^4\cdot c^{x-3}=c^y.$$

33. $$ax+by=c,\quad dx+cy=f.$$

126. Determinants of the second order. The arrangement of numbers $\begin{vmatrix}4 & 2\\ 5 & 3\end{vmatrix}$ has been given the meaning $4\cdot 3-5\cdot 2$.

Such an arrangement is called a **determinant.**

The value of any such determinant is easily found since $\begin{vmatrix}a & c\\ b & d\end{vmatrix}$ means $ad-bc$.

Accordingly, $\begin{vmatrix}5 & 6\\ 2 & 3\end{vmatrix}=5\cdot 3-2\cdot 6=15-12=3.$

Similarly, $\begin{vmatrix}6 & -2\\ 8 & 3\end{vmatrix}=6\cdot 3-8\cdot(-2)=18+16=34.$

And $3\begin{vmatrix}3 & 9\\ -4 & -5\end{vmatrix}=3[-15-(-36)]=3[-15+36]=63.$

The preceding operations can be reversed and the difference (or sum) of two products written as a determinant.

Thus $mn-rs$ can be written: $\begin{vmatrix}m & s\\ r & n\end{vmatrix}$, or $\begin{vmatrix}m & r\\ s & n\end{vmatrix}$, and in a number of other ways.

Similarly, $ab-k=ab-1\cdot k=\begin{vmatrix}a & k\\ 1 & b\end{vmatrix}=\begin{vmatrix}b & 1\\ k & a\end{vmatrix}$, etc.

EXERCISES

Find the value of the determinants:

1. $\begin{vmatrix} 4 & 1 \\ 3 & 5 \end{vmatrix}$.

2. $\begin{vmatrix} 6 & -2 \\ 8 & 1 \end{vmatrix}$.

3. $3\begin{vmatrix} 2 & 3 \\ 4 & -3 \end{vmatrix}$.

4. $\begin{vmatrix} -3 & 2 \\ 7 & 8 \end{vmatrix}$.

5. $3\begin{vmatrix} \frac{1}{2} & 6 \\ \frac{1}{3} & 8 \end{vmatrix}$.

6. $4\begin{vmatrix} -3 & 5 \\ 7 & 8 \end{vmatrix}$.

7. $\begin{vmatrix} 2a & -10b \\ 5a & 3b \end{vmatrix}$.

8. $\frac{5}{2a}\begin{vmatrix} a^3 & 0 \\ a^2 & 2a \end{vmatrix}$.

9. $\frac{7}{3}\begin{vmatrix} 3e & 0 \\ 9d & -14e^2 \end{vmatrix}$.

Write as a determinant:

10. $ax - cr$.

11. $mz - 3d$.

12. $\frac{8}{3} - ar$.

13. $hk - c$.

14. $ab + cd$.

15. $a - b$.

Find the value of the fractions:

16. $\dfrac{\begin{vmatrix} 5 & 1 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$. **17.** $\dfrac{\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}}$. **18.** $\dfrac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$. **19.** $\dfrac{\begin{vmatrix} c & (1-c) \\ 7c & (7-12c) \end{vmatrix}}{\begin{vmatrix} c & 3 \\ 7c & 36 \end{vmatrix}}$.

Write as the quotient of two determinants:

20. $\dfrac{st - cd}{3cx - 5r}$.

21. $\dfrac{a + \frac{10}{c}}{a^2 - 12}$.

22. $\dfrac{ax - 6}{2r - 5t}$.

23. $\dfrac{\frac{m}{3} - 7}{2m - 1}$.

24. $\dfrac{6x - hm}{3a - 3r}$.

25. $\dfrac{x^2 + 0}{\frac{a^2}{4} + \frac{15}{a}}$.

127. Solution by determinants. For the general linear system in two unknowns

$$\begin{cases} ax + by = c, & (1) \\ dx + ey = f, & (2) \end{cases}$$

$$x = \frac{ce - bf}{ae - bd} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \ (3), \quad \text{and} \quad y = \frac{af - cd}{ae - bd} = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}. \quad (4)$$

The determinant expressions for x and y in (3) and (4) can be used as formulas to solve any pair of linear equations in two unknowns. This method is particularly useful in the solution of linear equations with literal coefficients. The determinant forms can be easily remembered and written down at once if we observe carefully the following points:

I. *The determinants in the denominators are identical, and each is formed by the coefficients of x and y as they stand in the original equations (1) and (2).*

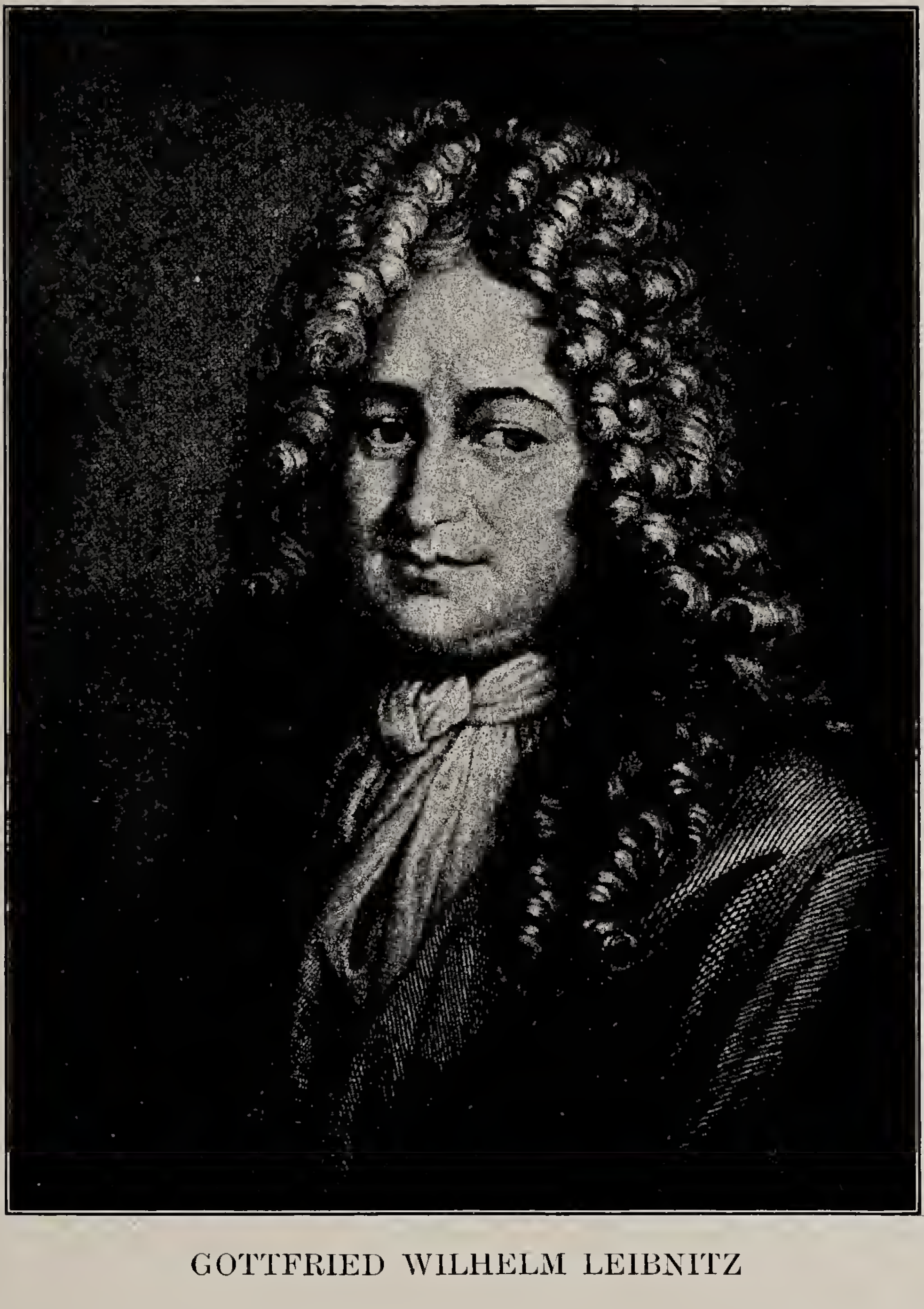
II. *The determinant in the numerator of the value of x is formed from the denominator by replacing the coefficients of x, $\begin{matrix} a \\ d \end{matrix}$, by the constant terms $\begin{matrix} c \\ f \end{matrix}$.*

III. *The determinant in the numerator of the value of y is formed from the denominator by replacing the coefficients of y, $\begin{matrix} b \\ e \end{matrix}$, by the constant terms $\begin{matrix} c \\ f \end{matrix}$.*

Biographical Note. Gottfried Wilhelm Leibnitz. For the last few hundred years the study of the higher mathematics has been carried on almost entirely by professors in the universities. It is rather exceptional for a man not connected with any educational institution to achieve distinction in this field. Before this was the case, however, scholars were accustomed to devote themselves to any or all branches of learning which attracted them, and many men of wide erudition in various walks of life flourished at different times during the two or three hundred years following the fifteenth century.

But of them all, the man who perhaps most clearly deserves the title of universal genius is Leibnitz (1646–1716). He was born in Leipzig, Germany, and on account of the poor instruction in the school to which he was sent, he was obliged to learn Latin by himself, which he did at the age of eight. By the time he was twelve he read Latin with ease, and had begun Greek. Not until the age of twenty-six, when he was sent to Paris on a political errand, did he become deeply interested in mathematics. From 1676, for nearly forty years, he held the well-paid position of librarian in the ducal palace of Brunswick, serving under three princes, the last of whom became George I of England in 1714. This post afforded him time for the deep study of mathematics, philosophy, theology, law, politics, and languages, in all of which he distinguished himself.

An incomplete edition of his mathematical works has been published in seven volumes. It is in his writings that we find the first mention of determinants. He also discovered the calculus independently of Sir Isaac Newton, and the last years of both men were embittered by a most unfortunate wrangle in which the friends of Newton accused Leibnitz of publishing as his own, results which really belonged to Newton.



GOTTFRIED WILHELM LEIBNITZ

Personally Leibnitz was quick of temper, impatient of contradiction, overfond of money, and one of the few really great men who have been offensively conceited.

Example: Solve by determinants $\begin{cases} 2y + x = 7, \\ 5x = 2y + 11. \end{cases}$

Solution: Writing the equations in the standard form, we have

$$x + 2y = 7,$$
$$5x - 2y = 11.$$

Then
$$x = \frac{\begin{vmatrix} 7 & 2 \\ 11 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix}} = \frac{-14-22}{-2-10} = \frac{-36}{-12} = 3.$$

In solving for y the denominator is the same as before; hence

$$y = \frac{\begin{vmatrix} 1 & 7 \\ 5 & 11 \end{vmatrix}}{-12} = \frac{11-35}{-12} = \frac{-24}{-12} = 2.$$

Check: $\begin{cases} 4 + 3 = 7, \\ 15 = 4 + 11. \end{cases}$

EXERCISES

Solve by determinants and check results:

1. $2x + 3y = 7,\quad 3x - 2y = 4.$
2. $4x = 3y + 8,\quad 5y + 6 = 3x.$
3. $5x + 4y = 10a + 4,\quad x - 2ay = 0.$
4. $.3x + .02y = 185,\quad .5x + .04y = 335.$
5. $4x + 3y = 6,\quad \dfrac{3x}{4} + \dfrac{3y}{4} = 3.$
6. $\dfrac{x}{2} - \dfrac{y}{3} = 6,\quad \dfrac{3x}{2} + \dfrac{2y}{3} = 8.$
7. $7x + 5y = 21c,\quad \dfrac{x}{c} - \dfrac{y}{2c} = 3.$
8. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a+b}{ab},\quad x - y = \dfrac{a^2 - b^2}{ab}.$

128. Indeterminate equations. If numerical values are given to any two variables in the equation $m + n + p = 6$, a value for the third variable can be found, which, taken with the values assigned to the other variables, satisfies the equation.

For example, let $m = 1$ and $n = 2$. Then $m + n + p = 6$ becomes $1 + 2 + p = 6$, whence $p = 3$. Obviously $m = 1$, $n = 2$, and $p = 3$ satisfy the equation. Other values may be given to m and n (or m and p, or n and p), and the foregoing process repeated, thus obtaining set after set of roots. A few sets of roots are tabulated here.

m	1	2	0	1	6	$\frac{1}{2}$	-4	10
n	2	2	0	3	0	$\frac{5}{2}$	2	-1
p	3	2	6	2	0	3	8	-3

a

It can easily be shown that the above table can be indefinitely extended; that is, that every linear equation in three variables has an infinite number of sets of roots.

It can also be shown that a system of two independent equations of the first degree in three variables has an infinite (unlimited) number of sets of roots.

A system of three *independent equations* of the first degree in three variables, no two equations being *incompatible*, has *one* set of roots and *only* one.

A system of four independent linear equations in three variables has no set of roots.

Note. It is not a little remarkable that the writings of the first great algebraist, Diophantos of Alexandria (about 300 A.D.), are devoted almost entirely to the solution of indeterminate equations; that is, to finding the sets of related values which satisfy an equation in two variables, or perhaps two equations in three variables. We know practically nothing of Diophantos himself, excepting the information contained in his epitaph, which reads as follows: "Diophantos passed one sixth of his life in childhood, one twelfth in youth, one seventh more as a bachelor; five years after his marriage a son was born who died four years before his father, at half his father's age." From this statement the reader was supposed to be able to find at what age Diophantos died. As a mathematician Diophantos stood alone, without any prominent forerunner, or disciple, so far as we know. His solutions of the indeterminate equations were exceedingly skillful, but the methods which he used were so obscure that his work had comparatively little influence upon that of later times.

129. Determinate systems. The method of obtaining the set of roots of a determinate system is illustrated in the example on page 222.

If necessary, the student should refer to that example and the rule and explanations on page 223.

EXERCISES

1. Find five sets of roots for $x - 2y + z = 6$.

2. Find three sets of roots for the system

$$\begin{aligned} m + n - p &= 8, \\ 3m - 2n + 4p &= 6. \end{aligned}$$

Solve the following systems:

3. $$\begin{aligned} 2x + 3y &= -14 - 4z, \\ x - y + 3z &= 0, \\ 5x + z &= 14 - 2y. \end{aligned}$$

4. $$\begin{aligned} x + 2y + 3z &= 14, \\ 4x - 5y + 6z &= 12, \\ x + 15y + 9z &= 58. \end{aligned}$$

5. $$\begin{aligned} .4r + .3s - 8t &= 4, \\ .5r + t + .8s &= 1.2, \\ 2.6t + .3 - r &= +.5s. \end{aligned}$$

6. $$\begin{aligned} .25x + .05y &= -1 + .10z, \\ .50x - .30y &= 0, \\ .05y + .04z &= 3. \end{aligned}$$

In Exercises 7 and 8 consider a, b, c as known numbers.

7. $$\begin{aligned} \frac{r}{a} + \frac{3s}{2a} - \frac{t}{3a} &= 6, \\ 7r + 4t &= 6s, \\ \frac{t}{9} - \frac{s}{6} &= 0. \end{aligned}$$

8. $$\begin{aligned} h + 2k - l &= 3b + c, \\ 5h - 4k - 4l &= a + b - 8c, \\ \frac{h}{2} &= \frac{3a + b}{6} + \frac{k}{3} - \frac{2c}{6}. \end{aligned}$$

In Exercises 9–11 solve for x, y, and z. Solve Exercise 15 for x only.

9. $$\begin{aligned} a^x \cdot a^{2y} \cdot a^z &= a^{-1}, \\ c^{2x} \cdot c^{-y} \cdot c^{z+1} &= c^{-19}, \\ e^{-x} \cdot e^{-y} \cdot e^{-5z} &= e^{13}. \end{aligned}$$

10. $$\begin{aligned} b^x \cdot b^{y+2} &= b^{12}, \\ c^x \cdot c^{z+4} &= c^{16}, \\ d^y \cdot d^{z+3} &= d^{17}. \end{aligned}$$

11. $$\begin{aligned} \frac{a}{x} + \frac{b}{y} + \frac{c}{z} &= 3, \\ \frac{a}{x} - \frac{b}{y} + \frac{2c}{z} &= 2, \\ \frac{5a}{x} - \frac{2c}{z} &= \frac{3b}{y}. \end{aligned}$$

12. $$\begin{aligned} r+s+t+u&=6,\\ 2r-s+3t-u&=-16,\\ 5r+9s+4u&=81+6t,\\ r+9t-7u&=-54-5s. \end{aligned}$$

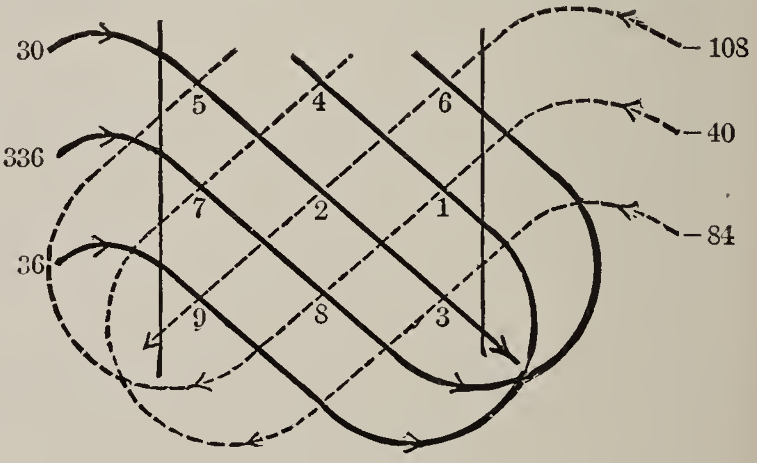
14. $$\begin{aligned} 4h-k+m&=0,\\ 7k+2m+x&=0,\\ 4m+x+8h&=0,\\ 16h+5k-x&=4. \end{aligned}$$

13. $$\begin{aligned} 4x-3y+2z&=20,\\ 5x+4y-10z&=3,\\ 34z-7x-18y&=31. \end{aligned}$$

15. $$\begin{aligned} ax+by+cz&=p,\\ dx+ey+fz&=q,\\ gx+hy+iz&=r. \end{aligned}$$

130. Determinants of the third order. The arrangement of numbers $\begin{vmatrix} 5 & 4 & 6\\ 7 & 2 & 1\\ 9 & 8 & 3 \end{vmatrix}$ has been given the meaning $5\cdot2\cdot3+7\cdot8\cdot6+9\cdot1\cdot4-6\cdot2\cdot9-1\cdot8\cdot5-3\cdot7\cdot4$, which equals $30+336+36-108-40-84=170$.

Such an arrangement is called a determinant of the **third order** because it has three **rows** (horizontal lines of numbers) and three **columns** (vertical lines of numbers). Each of the nine numbers in the determinant is called an **element**.



Every determinant of the third order is equal to a polynomial of six terms. Each of the six terms is the product of three elements so chosen that *one element, and only one, is taken from each row, and one element, and only one, is taken from each column.* If each element is positive, three terms of the polynomial are positive and three are negative. In connection with the preceding explanation the student should study carefully the above diagram, in which each continuous line connects three numbers whose product gives a positive term, and each dotted line connects three numbers whose product gives a negative term.

It follows, then, that the preceding determinant is equal to:

$$\begin{array}{ccccccc} 5\cdot2\cdot3 & +7\cdot8\cdot6 & +9\cdot1\cdot4 & -6\cdot2\cdot9 & -1\cdot8\cdot5 & -3\cdot7\cdot4 & = \\ 30 & +336 & +36 & -108 & -40 & -84 & =170. \end{array}$$

When finding the value of each product in a determinant, the sign of every negative element must be taken into account along with the foregoing explanations.

EXERCISES

Find the value of:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 4 & 1 \end{vmatrix}.$

2. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$

3. $\begin{vmatrix} 1 & -2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}.$

4. $\begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ a & b & c \end{vmatrix}.$

5. $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 4 & 5 \\ -3 & 1 & 1 \end{vmatrix}.$

6. $\begin{vmatrix} 1 & 1 & 1 \\ a & 1 & a \\ -a & 5 & 6 \end{vmatrix}.$

7. $-\begin{vmatrix} 2 & -1 & 3 \\ -3 & 1 & 2 \\ 4 & 5 & 1 \end{vmatrix}.$

8. $\begin{vmatrix} a & 2 & 7 \\ b & 3 & 8 \\ c & 4 & 9 \end{vmatrix}.$

9. $\begin{vmatrix} x & y & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 1 \end{vmatrix}.$

10. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$

11. $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$

12. $c\begin{vmatrix} 5 & 1 & b \\ c & 0 & 0 \\ 8 & 6 & a \end{vmatrix}.$

131. General linear system in three variables. For the system

$$ax + by + cz = p, \tag{1}$$

$$dx + ey + fz = q, \tag{2}$$

$$gx + hy + iz = r, \tag{3}$$

$$x = \frac{pei + qhc + rfb - cer - fhp - iqb}{aei + dhc + gfb - ceg - fha - idb}, \tag{4}$$

$$y = \frac{aqi + drc + gfp - cqg - fra - idp}{aei + dhc + gfb - ceg - fha - idb}, \tag{5}$$

$$z = \frac{aer + dhp + gqb - peg - qha - rdb}{aei + dhc + gfb - ceg - fha - idb}. \tag{6}$$

(See Exercise 15, page 328.)

In the fractions in (4), (5), and (6) observe the following points:

1. The three denominators are identical.

2. Each numerator and each denominator contains six terms, three positive and three negative.

3. Each term is the product of three letters, one of these letters, and only one, being taken from each equation.

4. Each term in the numerator differs from the term just below it in the denominator by one letter, and only one.

The facts just stated will help to make clear the reason for what now follows.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \tag{D}$$

Writing the coefficients of x, y, and z as a determinant in the order in which they occur in (1), (2), and (3), we obtain (D).

But $(D) = aei + dhc + gfb - ceg - fha - idb$, which is the denominator of the fractions in (4), (5), and (6).

If in (D) we now replace the coefficients of x, $\begin{matrix}\mathbf{a}\\ \mathbf{d},\\ \mathbf{g}\end{matrix}$ by the constant terms $\begin{matrix}\mathbf{p}\\ \mathbf{q},\\ \mathbf{r}\end{matrix}$ and expand, we obtain a determinant equivalent to the *numerator of the fraction* (4) *whose value is* x*;* for

$$\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix} = pei + qhc + rfb - cer - fhp - iqb.$$

Again, if in (D) we replace the coefficients of y, $\begin{matrix}\mathbf{b}\\ \mathbf{e},\\ \mathbf{h}\end{matrix}$ by the constant terms $\begin{matrix}\mathbf{p}\\ \mathbf{q},\\ \mathbf{r}\end{matrix}$ we obtain a determinant equivalent to the *numerator of the fraction* (5) *whose value is* y.

Lastly, if in (D) we replace the coefficients of z, $\begin{matrix}\mathbf{c}\\ \mathbf{f},\\ \mathbf{i}\end{matrix}$ by the constant terms $\begin{matrix}\mathbf{p}\\ \mathbf{q},\\ \mathbf{r}\end{matrix}$ we obtain a determinant equivalent to the *numerator of the fraction* (6) *whose value is* z. (The student should perform the work outlined in the last two sentences.)

Therefore we may write the values of x, y, and z for the given system in determinant form as follows:

$$x = \frac{\begin{vmatrix} \mathbf{p} & b & c \\ \mathbf{q} & e & f \\ \mathbf{r} & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad (7) \qquad y = \frac{\begin{vmatrix} a & \mathbf{p} & c \\ d & \mathbf{q} & f \\ g & \mathbf{r} & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad (8) \qquad z = \frac{\begin{vmatrix} a & b & \mathbf{p} \\ d & e & \mathbf{q} \\ g & h & \mathbf{r} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad (9)$$

The fractions (4), (5), and (6) are general results, and can be used as formulas to solve any three simultaneous equations in three variables, but the equivalent forms (7), (8), and (9) are far more easily remembered. These can be written down at once for *any* system of three equations in three variables, since

I. *The determinants in the denominators are identical, and each is formed by the coefficients of* x, y, *and* z, *as they stand in the original equations.*

II. *Each determinant in the numerator is formed from the denominator by putting the column of constant terms (as they stand in the original equation) in place of the column of the coefficients of the variable whose value is sought.*

The method of solution by determinants of a system of equations in three variables is illustrated in:

Example 1. Solve the system
$$\begin{cases} 3x + y = 14 - z, & (1)\\ x + z = 1 + 2y, & (2)\\ x + y = 15 - 2z. & (3)\end{cases}$$

Solution: Rewriting in standard form,
$$3x + y + z = 14, \tag{4}$$
$$x - 2y + z = 1, \tag{5}$$
$$x + y + 2z = 15. \tag{6}$$

From I and II preceding,
$$x = \frac{\begin{vmatrix} 14 & 1 & 1\\ 1 & -2 & 1\\ 15 & 1 & 2\end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & 2\end{vmatrix}} = \frac{-26}{-13} = 2, \text{ and } y = \frac{\begin{vmatrix} 3 & 14 & 1\\ 1 & 1 & 1\\ 1 & 15 & 2\end{vmatrix}}{-13} = \frac{-39}{-13} = 3.$$

The value of z can now be more easily obtained by substituting the values of x and y already found in (1), (2), or (3) than by means of determinants.

Substituting in (2), $\quad 2 + z = 1 + 6;$
whence $\quad z = 5.$

Check: (1) + (2) + (3) gives
$$5x + 2y + z = 30 + 2y - 3z.$$
Substituting, $\quad 10 + 6 + 5 = 30 + 6 - 15,$
or $\quad 21 = 21.$

Example 2. Solve the system
$$\begin{cases} x + y = 13 + 2z, & (1)\\ x + 7 = 3y, & (2)\\ x + 4z = -14. & (3)\end{cases}$$

Solution: Rewriting in standard form and supplying zero coefficients,
$$x + y - 2z = 13, \tag{4}$$
$$x - 3y + 0z = -7, \tag{5}$$
$$x + 0y + 4z = -14. \tag{6}$$

Then
$$x = \frac{\begin{vmatrix} 13 & 1 & -2\\ -7 & -3 & 0\\ -14 & 0 & 4\end{vmatrix}}{\begin{vmatrix} 1 & 1 & -2\\ 1 & -3 & 0\\ 1 & 0 & 4\end{vmatrix}} = \frac{-44}{-22} = 2.$$

The value of y can now be more easily obtained by substituting 2 for x in (2) than by means of determinants.

Accordingly $$2 + 7 = 3y;$$
whence $$y = 3.$$

Similarly, by substituting 2 for x in (3), $2 + 4z = -14;$
whence $$z = -4.$$

Check: (1) + (2) + (3), $3x + y + 4z + 7 = -1 + 2z + 3y.$
Substituting, $6 + 3 - 16 + 7 = -1 - 8 + 9$, or $0 = 0.$

Note. As we have seen, determinants have a very useful application in the solution of systems of linear equations in two or three variables. With some practice one can solve such equations more rapidly by determinants than by the other methods which have been given. If the student studies advanced algebra, he will learn of determinants of the fourth and higher orders, and of the usefulness of such determinants in solving linear systems in four or more variables. Moreover, he will then see that the theory of determinants is an absolute necessity for the discussion of the general theory of linear systems in n variables.

EXERCISES

Solve for x, y, and z as in the two preceding examples:

1. $$\begin{gathered} x + y + z = 1,\\ x + y - z = 2,\\ x - y + z = 3. \end{gathered}$$

2. $$\begin{gathered} x + 2y + z = 1,\\ 2x + y - z = 0,\\ x + 2y - z = 0. \end{gathered}$$

3. $$\begin{gathered} 2x + y = 5 + z,\\ x - 2z = 6,\\ 3y + 2z = x. \end{gathered}$$

4. $$\begin{gathered} x + y = 1,\\ x + z = 2,\\ y + z = 3. \end{gathered}$$

5. $$\begin{gathered} x + y = 3a,\\ x + z = 4a,\\ y + z = 5a. \end{gathered}$$

6. $$\begin{gathered} \frac{x}{3} + \frac{y}{2} = 9,\\ \frac{x}{2} + \frac{z}{3} = 8,\\ \frac{y}{3} + \frac{z}{2} = 13. \end{gathered}$$

7. $$\begin{gathered} ax + by = 0,\\ cx - bz = 2bc,\\ bx + az - cy = b^2. \end{gathered}$$

8. $$\begin{gathered} hx + ky - lz = 2hk,\\ ky - hx + lz = 2kl,\\ hx - ky + lz = 2hl. \end{gathered}$$

9. $$\begin{gathered} mx + mx_1 + x_2 = 0,\\ mx + x_1 + mx_2 = ma - a,\\ mx - 3mx_1 + x_2 = 4ma. \end{gathered}$$

Note. Like so many other discoveries, the determinant notation was noticed independently by two men. In a letter to a friend, written in 1693, Leibnitz outlined the method of solving equations by the means of determinants; but, so far as we know, he used the notation in his own work very little, and certainly did not publish it during his lifetime. In fact, the letter in which this reference is found, did not come to light until 1850, and the fact that Leibnitz knew anything about determinants was not generally recognized until after that time.

In 1750 Cramer, a professor in the university at Geneva, rediscovered this method of solving linear systems; and his work had the good fortune to be accepted by scholars, forming the real beginning of the development of the subject. Since that time a great many have written on the subject, and to-day determinants are used in every field of advanced mathematics.

PROBLEMS

1. A and B together can do a piece of work in $3\frac{3}{7}$ days. If they work together 2 days and A can then finish the job alone in $2\frac{1}{2}$ days more, how many days does each require alone?

2. A man and a boy can do in 18 days a piece of work which 5 men and 9 boys can do in 3 days. In how many days can one man do the work? one boy?

3. If $ax + by = 2$ is satisfied by $x = 2$ and $y = 3$, and also by $x = 6$ and $y = 5$, what values must a and b have?

4. A launch, whose rate in still water is 12 miles per hour, goes up a stream whose rate is 2 miles per hour, and returns. The entire trip requires 24 hours. Find the number of hours required for the trip upstream and the number for the return.

5. Two sums are put at interest at 5% and 6% respectively. The annual income from both together is \$100. If the first sum had yielded 1% more and the second 1% less, the annual income would have been decreased \$2. Find each sum.

6. A sum of \$4000 is invested, a part in 5 per cent bonds at 90, and the remainder in 6 per cent bonds at 110. If the total annual income is \$220, find the sum invested at each rate.

7. A train leaves M two hours late and runs from M to P at 50% more than its usual rate, arriving on time. If it had run from M to P at 25 miles per hour, it would have been 48 minutes late. Find the usual rate and the distance from M to P.

8. A train leaves M thirty minutes late. It then runs to N at a rate 20% greater than usual, and arrives 6 minutes late. Had it run 15 miles of the distance from M to N at the usual rate and the rest of the trip at the increased rate, it would have been 12 minutes late. Find the distance from M to P and the usual rate of the train.

9. The length of a freight train is 1430 feet and the length of a passenger train 550 feet. When they run on parallel tracks in opposite directions they pass each other in 18 seconds, and when they run in the same direction they pass each other in 1 minute and 30 seconds. Find the rates of the trains.

10. Two contestants run over a 440-yard course. The first wins by 4 seconds when given a start of 200 feet. They finish together when the first is given a handicap of 40 yards. Find the rate of each in feet per second.

11. It is desired to have a 10-gallon mixture of 45% alcohol. Two mixtures, one of 95% alcohol and another of 15% alcohol, are to be used. How many gallons of each will be required to make the desired mixture?

12. The crown of Hiero of Syracuse, which was part gold and part silver, weighed 20 pounds, and lost $1\frac{1}{4}$ pounds when weighed in water. How much gold and how much silver did it contain if $19\frac{1}{4}$ pounds of gold and $10\frac{1}{2}$ pounds of silver each lose one pound when weighed in water?

13. One angle of a triangle is twice another, and their sum equals the third. Find the number of degrees in each angle of the triangle.

14. The sum of three numbers is 217. The quotient of the first by the second is 5, which is also the quotient of the second by the third. Find the numbers.

15. If the tens' and units' digits of a 3-digit number be interchanged, the resulting number is 27 less than the given number. If the same interchange is made with the tens' and hundreds' digits, the resulting number is 180 less than the given number. The sum of the digits is 14. Find the number.

16. In one hour a tank which has three intake pipes is filled seven eighths full by all three together. The tank is filled in $1\frac{1}{3}$ hours if the first and second pipes are open, and in 2 hours and 40 minutes if the second and third pipes are open. Find the time in hours required by each pipe to fill the tank.

17. The sums of three pairs of adjacent sides of a quadrilateral are respectively 80 feet, 108 feet, and 116 feet. The difference of the fourth pair of adjacent sides is 24 feet. Find each side.

18. Two chairs cost h dollars. The first cost m cents more than the second. Find the cost of each in cents.

19. A and B together have d dollars. A gives c dollars to B, after which B gives m dollars to A. Then A has $\frac{1}{3}$ as many dollars as B. Find the number of dollars each had at first.

20. A and B together can do a piece of work in m days. B works c times as fast as A. How many days does each require alone?

21. A man rows m miles downstream in t hours and returns in a hours. Find his rate in still water and the rate of the river.

22. A man dying leaves a widow and five children. The law provides that the widow shall receive one half of the estate and that the other half shall be divided equally among the children. The executor of the estate, after paying all debts, has $1200 in cash. But two of the children had borrowed from their father $400 each, for which he had accepted their notes. The executor found these notes worthless. How should he divide the cash on hand?

23. Solve in positive integers $5x + 2y = 42$.

Hint. $x = \frac{42 - 2y}{5} = 8 + \frac{2 - 2y}{5}$. Now if x is to be integral, $\frac{2 - 2y}{5}$ must be integral or zero; that is, $2 - 2y$ must be zero or an integral multiple of 5. Hence the least value of y is 1.

The various related sets of values which satisfy this equation may be effectively represented to the eye by the graph of the equation. Then if the line whose equation is $5x + 2y = 42$ passes through any points both of whose coördinates are positive integers, each pair of these values is a set of roots. If the line does not enter the first quadrant, we can see at a glance that the equation has no set of roots which are positive integers.

24. Solve in positive integers $7x + 2y = 36$, and illustrate the result graphically.

25. In how many ways can a debt of $73 be paid with five-dollar and two-dollar bills? Illustrate the result graphically.

26. A man buys calves at $6 each and pigs at $4 each, spending $72. How many of each did he buy?

27. In how many ways can $1.75 be paid in quarters and nickels?

28. A farmer sells some calves at $6 each, pigs at $3 each, and lambs at $4 each, receiving for all $126. In how many ways could he have sold 32 animals at these prices for the same sum? Determine the various groups.

29. In how many ways can a sum of $2.40 be made up with nickels, dimes, and quarters, on the condition that the number of nickels used shall equal the number of quarters and dimes together? Determine the various groups.

CHAPTER XXVIII

ROOTS AND RADICALS

(In Part Review)

132. Roots of algebraic expressions. The student should make any necessary review of pages 101–103 and 228–238.

EXERCISES

1. State the rule for the sign of (*a*) the odd root of a number; (*b*) the even root of a positive number.

2. State the rule for extracting the fourth root of a monomial.

3. State the rule for extracting the fifth root of a monomial.

4. How can one obtain the fourth root of a polynomial?

5. State the rule for extracting the square root of a polynomial.

6. What is the value of $\sqrt{4}$? of $\sqrt[4]{81}$?

7. Can one obtain the fifth root of a number (*a*) by extracting the square root of its cube root? (*b*) by extracting the cube root of its square root? Explain.

Extract the square roots of:

8. $a^6 - 10a^4 - 4a^3 + 25a^2 + 20a + 4.$

9. $4a^8 + 12a^4 - 7 - 24a^{-4} + 16a^{-8}.$

10. $49c^{-6} - 28c^{-4} + 74c^{-2} - 20 + 25c^2.$

11. $9x^4 - 6x^{\frac{7}{2}} + x^3 - 66x^{\frac{5}{2}} + 22x^2 + 121x.$

12. $16m^{-7} - 8m^{-4} + 104m - 26m^4 + 169m^9 + m^{-1}.$

13. $\dfrac{a^2}{9c^2} + \dfrac{8c^2}{a^2} + \dfrac{2a}{c} + \dfrac{8c}{a} + \dfrac{10}{3} + \dfrac{8c^2}{a^2} + \dfrac{8c}{a} + \dfrac{10}{3} + \dfrac{2a}{3c}.$

14. $\frac{1}{4a^4} - \frac{3x}{a^3} + \frac{9x^2}{a^2} + \frac{a^4}{25x^2} - \frac{6a}{5} + \frac{1}{5x}$.

Find the first four terms in the square roots of:

15. $1 + 2x$. **16.** $\frac{25}{9} + a^3$.

17. Find the first three terms in the fourth roots of the expression in Exercise 10.

133. Graphical method of extracting roots. When obtaining the square roots of arithmetical numbers by the graphical method we proceed as follows: Let x represent any number and y the square of that number; that is, we let $y = x^2$. Then we construct the graph of this equation, obtaining first the table:

x	$\frac{1}{2}$	1	2	3	4	5	6	7	8
y	$\frac{1}{4}$	1	4	9	16	25	36	49	64

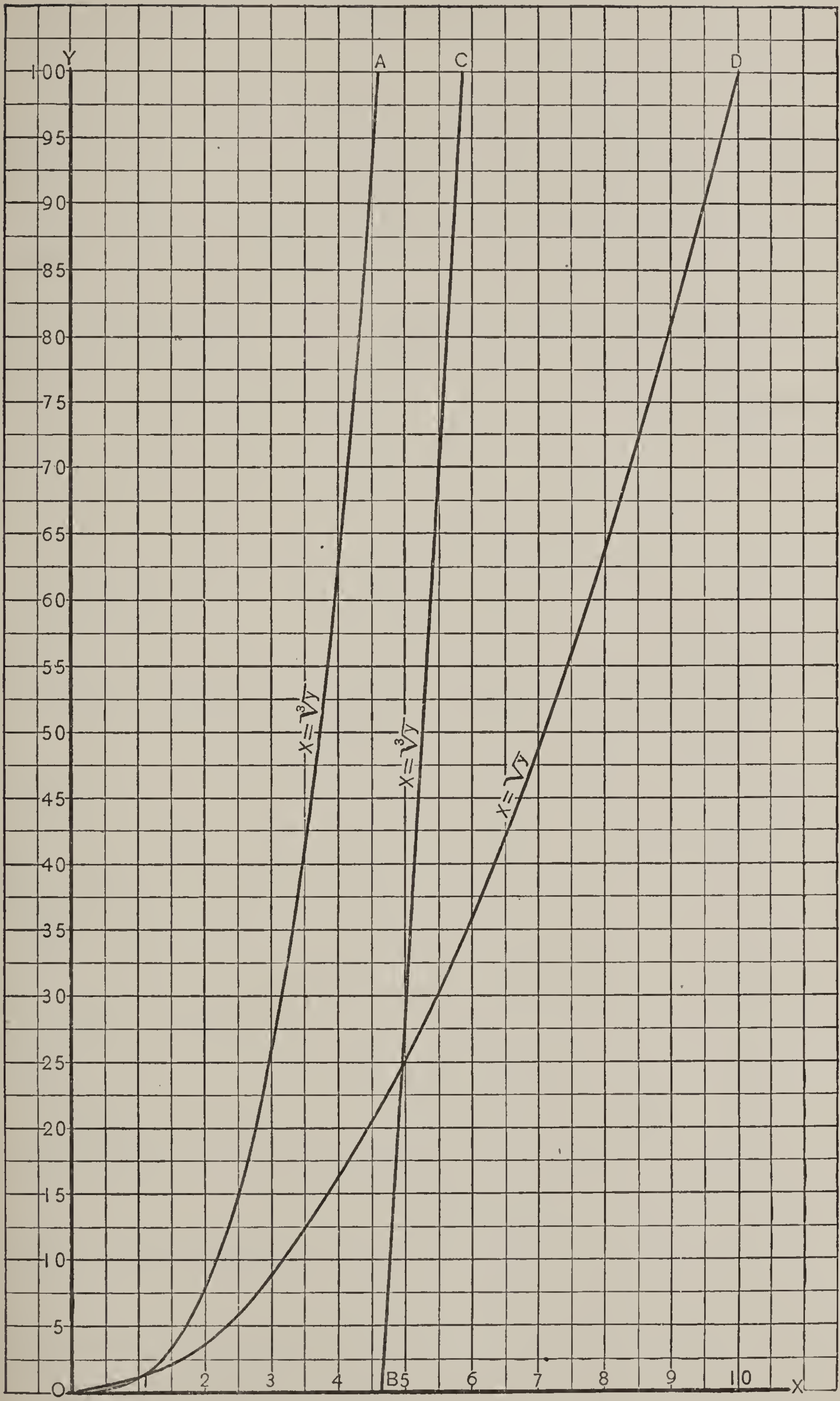
Plotting these values, we obtain curve OD of page 339.

From this curve we can read off the square root of any number between 1 and 100 correct to one decimal place.

Curve OA is a portion of the graph of $y = x^3$, and BC is a continuation of OA. From this curve we can obtain the cube root of any number between 1 and 200 correct to one decimal place. The cube root of numbers between 1 and 100 we obtain from curve OA; for numbers between 100 and 200 we obtain the cube root from BC.

If one desires greater precision or a larger range of numbers, or both, he can obtain them by using a large piece of cross-section paper and a different scale. Such a curve, if carefully drawn, is convenient for any computation not requiring too great accuracy. The point in its favor is that one can read off the square roots or the cube roots more rapidly than he can obtain them by the methods of §§ 134 and 219, or even by logarithms.

The graphical method can also be used to extract fourth and higher roots.


Y
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20
15
10
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O
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2
3
4
B
5
6
7
8
9
10
X
$x=\sqrt[3]{y}$
$x=\sqrt[3]{y}$
$x=\sqrt{y}$

EXERCISES

From the graph read:

1. The square root of (*a*) 20; (*b*) 45; (*c*) 59; (*d*) 68.

2. The cube root of (*a*) 25; (*b*) 18; (*c*) 52; (*d*) 165; (*e*) 150.

3. The square of (*a*) 2.4; (*b*) 6.1; (*c*) 7.9; (*d*) 8.3.

4. The cube of (*a*) 3.2; (*b*) 3.9; (*c*) 2.8; (*d*) 5.6; (*e*) 4.8.

134. Square roots of arithmetical numbers. The student should make any necessary review of the abbreviated process of extracting a square root of an arithmetical number given on page 233.

EXERCISES

Find the square roots of:

1. 6889.
2. 56169.
3. .6724.
4. 1.4641.
5. 4.2025.
6. .04028049.

Extract the square roots, correct to four decimal places, of:

7. 5.
8. .07.
9. $\frac{13}{7}$.
10. $\frac{237}{93}$.

11. Find the hypotenuse of the right triangle whose legs are 183 and 264 respectively.

12. A baseball diamond is a square 90 feet on each side. Find the distance from the home plate to second base, correct to .01 of a foot.

13. The hypotenuse of a right triangle is 207 feet and one leg is 83 feet. Find the other leg, correct to .01 of a foot.

14. The hypotenuse and one leg of a right triangle are respectively 292849 and 207000. Find the other leg.

15. The side of an equilateral triangle is 11 inches. Find its altitude, correct to .1 of an inch.

16. Find the side of an equilateral triangle whose altitude is 10 inches, correct to .001 of an inch.

17. Find the area of a triangle whose sides are 12, 27, and 35 inches respectively, correct to .001 of a square foot.

Fact from Geometry. If a, b, and c represent the sides of a triangle, and s equals one half of $a + b + c$, the area of the triangle equals $\sqrt{s(s-a)(s-b)(s-c)}$.

18. By the method of Exercise 17 find, correct to .01 of a square inch, the area of a triangle each side of which is 22 inches.

19. Find the radius of a circle whose area is 40 square feet.

20. Find the diagonal of a room whose dimensions in feet are 14, 20, and 30.

21. Find the diagonal of a cube whose edge is 1 foot.

22. A room is 24 feet by 40 feet by 14 feet. What is the length of the shortest broken line from one lower corner to the diagonally opposite upper corner, the line to be in part on the walls or the floor, but not through the air?

23. Take any two integers and form three others from them thus: find the sum of their squares, the difference of their squares, and twice their product. Is the square of one of the three resulting numbers equal to the sum of the squares of the other two? Discuss this with reference to the sides of a right triangle.

24. One leg of a right triangle is 28. Find all possible integral values for the other two sides.

135. Classification of numbers. All the numbers of algebra are in one or the other of two classes, **real** numbers and **imaginary,** or **complex** numbers.

Real numbers are of two kinds, **rational** numbers and **irrational** numbers.

A **rational** number is a positive or a negative *integer*, or a number which may be expressed as the *quotient of two such integers.* Any real number which is not rational, is **irrational.**

A **pure imaginary** number is the indicated square root of a negative number. An **imaginary,** or **complex,** number, when reduced to its simplest form, is the indicated sum of a real number and a pure imaginary.

136. Radicals. A **radical** is an indicated root of the form $\sqrt[r]{n}$ or $c\sqrt[r]{n}$.

A **surd** is an irrational root of a rational number.

The **index** determines the **order** of the radical and the root to be extracted.

The **radicand** is the number, or expression, under the radical sign.

Radical expressions may be written in two ways, with radical signs or with fractional exponents.

The student should make any necessary review of pages 239–240.

EXERCISES

Write with radical signs:

1. $x^{\frac{3}{5}}$. **4.** $4x^{\frac{1}{3}}$. **7.** $3a^{\frac{1}{2}}(c^2x)^{\frac{2}{3}}$.
2. $(ac)^{\frac{2}{3}}$. **5.** $6cx^{\frac{3}{4}}$. **8.** $\dfrac{4a^{\frac{2}{3}}(c-x)^{\frac{4}{3}}}{a^{\frac{1}{2}}(c-x)}$.
3. $(4x)^{\frac{1}{3}}$. **6.** $a^{\frac{2}{3}}c^{\frac{3}{5}}$.

Find the numerical value of:

9. $4^{\frac{1}{2}}$. **13.** $4^{\frac{3}{2}}$. **17.** $(-32)^{\frac{3}{5}}$.
10. $36^{\frac{1}{2}}$. **14.** $27^{\frac{2}{3}}$. **18.** $(\frac{4}{25})^{\frac{1}{2}}\cdot(\frac{125}{8})^{\frac{1}{3}}$.
11. $64^{\frac{1}{3}}$. **15.** $(-8)^{\frac{2}{3}}$. **19.** $(-125)^{\frac{2}{3}}\cdot(\frac{1}{25})^{\frac{3}{2}}$.
12. $81^{\frac{1}{4}}$. **16.** $(\frac{1}{9})^{\frac{1}{2}}$. **20.** $(-243)^{\frac{2}{5}}\cdot(81)^{\frac{3}{4}}$.

Write with fractional exponents:

21. $\sqrt[3]{x^4}$. **25.** $5\sqrt[3]{8x^4}$. **29.** $\dfrac{5c\sqrt{ax^3}\cdot\sqrt[3]{5x^4}}{\sqrt[3]{5x^2}\sqrt{ax}}$.
22. $\sqrt{ac^3}$. **26.** $6\sqrt[3]{64x^5}$.
23. $3\sqrt{x^5}$. **27.** $5\sqrt[3]{-125c^2}$. **30.** $\sqrt[n]{x^a}\cdot\sqrt[n]{c}$.
24. $4\sqrt[3]{4x^2}$. **28.** $c\sqrt{(x+a)^3}$. **31.** $\sqrt[n]{x^{4a}}\cdot\sqrt[n]{x^{2a}}$.

32. Give an example of (*a*) a real number; (*b*) an imaginary number; (*c*) a rational number; (*d*) an irrational number; (*e*) a radical; (*f*) a surd; (*g*) an index; (*h*) a radicand; (*i*) the principal odd root of a positive number; (*j*) the principal even root of a positive number; (*k*) the principal odd root of a negative number.

33. What is the distinction between a rational number and an irrational one?

34. Which of the numbers 8, $\frac{2}{3}$, $.\dot{3}4\dot{3}$, $\sqrt{4}$, $\sqrt{3}$, and π ($\pi = 3.14159 +$) are rational? irrational?

35. Give a geometrical illustration of an irrational number by means of a right triangle.

36. Is a radical always a surd? Illustrate.

37. Is a surd always a radical? Illustrate.

38. Distinguish between a surd and a radical.

39. Which of the numbers $\sqrt{3}$, $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt{\sqrt[3]{6}}$, $\sqrt{2+\sqrt{3}}$, and $\sqrt{\pi}$ are surds? Which are radicals?

40. What is the principal root of: $\sqrt{4}$, $\sqrt[3]{8}$, and $\sqrt[3]{-8}$?

41. Name the order of: $\sqrt{6}$, $a^{\frac{1}{3}}$, $\sqrt[3]{5}$, $c^{\frac{2}{3}}$, and $\sqrt[4]{m^3}$.

42. How many real numbers can be found for a designated odd root of (*a*) a positive real number? (*b*) a negative real number?

43. Change the word "odd" in (*a*) of Exercise 42 to "even," and answer.

137. Simplification of radicals. The form of a radical expression may be changed without altering its numerical value. It is often desirable to change the form of a radical so that its numerical value can be computed with the least possible labor. See pages 241 and 242 for explanations and examples.

EXERCISES

Express in simplest form:

1. $\sqrt{18}$.

2. $\sqrt[3]{16}$.

3. $2\sqrt{75}$.

4. $4\sqrt[3]{-54}$.

5. $\sqrt{\sqrt{4}}$.

6. $\sqrt[3]{3\sqrt{9}}$.

7. $\sqrt{\frac{3}{5}}$.

8. $\sqrt{\frac{5}{8}}$.

9. $\sqrt[3]{-\frac{3}{4}}$.

10. $\sqrt[3]{\frac{1}{7}}$.

11. $6\sqrt[3]{-\frac{1}{9}}$.

12. $\sqrt{3^2-(\frac{3}{2})^2}$.

13. $\sqrt{7^2 - (\frac{7}{2})^2}$.

14. $\sqrt{4 - 8\sqrt{3}}$.

15. $\sqrt{36 + 18\sqrt{5}}$.

16. $\sqrt[3]{81 - 3\sqrt{243}}$.

17. $\sqrt{R^2 - 3R^2\sqrt{5}}$.

18. $\sqrt{\dfrac{R^2 - R^2\sqrt{6}}{3}}$.

19. $\sqrt{R^2 - \left(\dfrac{R}{2}\right)^2}$.

20. $\sqrt{R^2 + \left(\dfrac{R}{3}\right)^2\sqrt{3}}$.

Express entirely under the radical sign:

21. $3\sqrt{5}$. **22.** $2\sqrt[3]{8}$. **23.** $2c\sqrt[3]{c^2}$. **24.** $4\sqrt[3]{\frac{1}{4}}$. **25.** $\dfrac{a}{3}\sqrt[3]{\dfrac{9}{a^2}}$.

26. $(2a+1)\sqrt{\dfrac{2}{4a^2-1}}$. **27.** $\dfrac{x-3a}{5}\sqrt[3]{\dfrac{125}{(x-3a)^2}}$.

28. Show that $\dfrac{c}{2}\sqrt{a^2 - \left(\dfrac{a^2+c^2-b^2}{2c}\right)^2} = \sqrt{s(s-a)(s-b)(s-c)}$,
if $a+b+c = 2s$.

Express in simplest form with one radical sign:

29. $\sqrt{\sqrt{x}}$.

30. $\sqrt{\sqrt[3]{x}}$.

31. $\sqrt[3]{\sqrt{x}}$.

32. $\sqrt[3]{\sqrt{8a^2x}}$.

33. $\sqrt{3\sqrt{3}}$.

34. $\sqrt{3\sqrt{3\sqrt{3}}}$.

35. $2\sqrt[3]{2\sqrt[3]{2}}$.

36. $\sqrt[a]{\sqrt[n]{x^c}}$.

37. $\sqrt[\frac{m}{n}]{x^{\frac{a}{n}}}$.

138. Addition and subtraction of radicals. **Similar** radicals are radicals of the same order with radicands which are identical or which can be made so by simplification.

The sum or the difference of similar radicals can be expressed as one term, while the sum or difference of dissimilar radicals can only be indicated.

EXERCISES

Simplify and collect:

1. $\sqrt{50} + \sqrt{98} - \sqrt{32}$.
2. $\sqrt[3]{192} - 4\sqrt[3]{24} + \sqrt[3]{375}$.
3. $10\sqrt{\frac{6}{5}} - \sqrt{\frac{3}{10}} + 4\sqrt{\frac{15}{2}}$.
4. $3\sqrt{\frac{2}{7}} + 3\sqrt{\frac{7}{2}} - 2\sqrt{\frac{1}{14}}$.
5. $\sqrt{x^3} + \sqrt[4]{x^2} - 12\sqrt[6]{x^3}$.
6. $\sqrt{\dfrac{3a}{x}} + \sqrt{\dfrac{3x}{a}} - \sqrt{\dfrac{ax}{3}}$.
7. $\sqrt{\dfrac{a}{x^3}} - \sqrt{\dfrac{a}{x^5}} + \sqrt{\dfrac{5x^3}{a}}$.
8. $\sqrt[4]{32x^5} + \sqrt[4]{1250x} - \sqrt[4]{512x} - \sqrt[4]{2x}$.

9. $\sqrt[3]{(a-c)^4} + c\sqrt[6]{a^2-2ac+c^2} + (a+c)\sqrt[3]{a-c}.$

10. $\sqrt{\frac{a}{c}} - \sqrt{\frac{c}{a}} + \sqrt{\frac{a^2+c^2}{ac}+2} - \sqrt{\frac{a^2+c^2}{ac}-2}.$

11. $\sqrt[3]{24} + \sqrt[3]{(3a+9)(a+3)^2} - \sqrt[3]{81} + a\sqrt[6]{9} - 4\sqrt[3]{3}.$

12. $2\sqrt{9a^3-9a^2b} - 3\sqrt{9ab^2-9b^3} + \sqrt{(a^2-b^2)(a+b)}.$

13. $(a-b)\sqrt{\frac{a+b}{a-b}} + \sqrt{25a^2-25b^2} + \frac{a+b}{a-b}\sqrt{\frac{36ab^2-36b^3}{a+b}}.$

139. Multiplication of real radicals. Real radicals of the same order are multiplied as explained in § 94.

EXERCISES

Perform the indicated multiplications and simplify the products:

1. $\sqrt{3}\cdot\sqrt{27}.$
2. $\sqrt{\frac{4}{5}}\cdot\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{3}{4}}.$
3. $\sqrt{3}\cdot\sqrt[3]{2}.$
4. $\sqrt[4]{6}\cdot\sqrt{2}.$
5. $\sqrt[3]{12}\cdot\sqrt{\frac{1}{8}}.$
6. $\sqrt[3]{a^2}\cdot\sqrt{a^3}.$
7. $\sqrt[3]{\frac{x}{a}}\cdot\sqrt{\frac{a}{x}}.$
8. $\sqrt[4]{2x^3}\cdot\sqrt{3x}.$
9. $(\sqrt{x-3})^2.$
10. $(2\sqrt{3x-1})^2.$
11. $(3\sqrt[4]{x-2})^2.$
12. $3\sqrt{x-3}\cdot\sqrt{4x-8}.$
13. $(\sqrt{x}-\sqrt{x-3})^2.$
14. $(\sqrt{x-3}-\sqrt{4x-7})^2.$
15. $(\sqrt{x}-\sqrt{3x})(4\sqrt{x}).$
16. $\left(\frac{6-2\sqrt{5}}{3}\right)^2\cdot\frac{[(\sqrt{5}+1)(\sqrt{5}+1)]^2}{2}\cdot\frac{(\sqrt{2}+1)(9\sqrt{2}-9)}{16}.$

Arrange in order of magnitude:

17. $\sqrt[3]{11},\ \sqrt{5}.$

Solution: $\sqrt[3]{11} = 11^{\frac{1}{3}} = 11^{\frac{2}{6}} = \sqrt[6]{11^2} = \sqrt[6]{121}.$

$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}. \quad \therefore \sqrt{5} > \sqrt[3]{11}.$

18. $\sqrt[3]{6},\ \sqrt{3}.$
19. $\sqrt[3]{19},\ \sqrt{7}.$
20. $\sqrt{3},\ \sqrt[4]{6}.$
21. $2\sqrt{5},\ \sqrt[3]{89}.$

22. $3\sqrt{3},\ 2\sqrt[3]{10}.$

23. $\sqrt[4]{48},\ \sqrt[5]{64}.$

24. $\sqrt{3},\ \sqrt[3]{6},\ \sqrt[9]{125}.$

25. $4\sqrt[4]{6},\ 3\sqrt[6]{25},\ 4\sqrt[9]{64}.$

Reduce to respectively equivalent surds of the same order:

26. $\sqrt{3},\ \sqrt[3]{3x^2}.$

27. $\sqrt[3]{a+b},\ \sqrt{a-b}.$

28. $2x\sqrt[3]{5xy},\ 5x\sqrt[4]{3xy}.$

29. $\sqrt[3]{xy},\ \sqrt[4]{xy^2},\ \sqrt[5]{x^2y}.$

Square:

30. $\sqrt[3]{3}.$

31. $2\sqrt[3]{4}.$

32. $\sqrt{5-\sqrt{5}}.$

33. $4\sqrt{3-\sqrt{5}}.$

34. $\sqrt[3]{4-4\sqrt{3}}.$

35. $\sqrt[4]{6-3\sqrt{2}}.$

Cube:

36. $3\sqrt{5}.$

37. $3\sqrt[3]{2}-2\sqrt{3}.$

38. $\left(\sqrt[4]{1-\sqrt{2}}\right)^3.$

Simplify:

39. $(5\sqrt{5}+9\sqrt{3}-\sqrt{7}+2\sqrt{105})(\sqrt{3}+\sqrt{5}-\sqrt{7}).$

40. $(\sqrt{a}-\sqrt[4]{ac}+\sqrt{c})(\sqrt{a}+\sqrt[4]{ac}+\sqrt{c})(a+c+\sqrt{ac}).$

41. $(\sqrt{2x-1}-\sqrt{5})(2\sqrt{2x-1}+\sqrt{45})(4x-\sqrt{10x-5}-17).$

42. $\sqrt{R^2-\left(\frac{R}{3}\sqrt{5}-\frac{R}{3}\right)^2}.$

43. $\left[\left(R-\frac{R}{6}\sqrt{3}\right)^2+\left(\frac{R}{6}\right)^2\right]^{\frac{1}{2}}.$

44. $\sqrt{R^2-\left(\frac{R\sqrt{4-\sqrt{2}}}{2}\right)^2}.$

45. $\left(\frac{R}{2}\sqrt{3+\sqrt{3}}\right)\left(\frac{R\sqrt{3-\sqrt{3}}}{2}\right).$

46. $\sqrt{\left(e^x+\frac{2}{e^x}\right)^2-(e^x-2e^{-x})(e^x-2e^{-x})+e^{2x}+e^{-2x}-6}.$

140. Division of real radicals. Division of one real radical by another may often be performed as in Examples 1–3, page 251.

Direct division of radical expressions in which the divisor is a polynomial is very difficult. Where division by a polynomial divisor is necessary we use the rule of pages 252–253.

This rule applies in all cases, while the rule for direct division fails when dividing a real radical by a radical of even order whose radicand is negative.

EXERCISES

Find a simple rationalizing factor for:

1. $3\sqrt{7}$.

2. $5\sqrt[3]{4}$.

3. $7\sqrt[4]{8}$.

4. $\sqrt[3]{16}$.

5. $\sqrt{5}-7$.

6. $3\sqrt{7}-2\sqrt{13}$.

7. $\sqrt{3a}-\sqrt{2x}$.

8. $\sqrt{x-c}-\sqrt{a}$.

9. $\sqrt{2}+\sqrt{3}-\sqrt{5}$.

Perform the indicated division and simplify results:

10. $\sqrt{8}\div\sqrt{24}$.

11. $8\div 4\sqrt{3}$.

12. $24\div 3\sqrt{3}$.

13. $a\div c\sqrt{x}$.

14. $(\sqrt{12}-\sqrt{18})\div 2\sqrt{3}$.

15. $(12-3\sqrt{6}-4\sqrt{24})\div 3\sqrt{2}$.

16. $\sqrt{6}\div\sqrt[3]{2}$.

HINT. $\sqrt{6}\div\sqrt[3]{2}=\dfrac{\sqrt{6}}{\sqrt[3]{2}}=\dfrac{\sqrt{6}\,\sqrt[3]{4}}{\sqrt[3]{2}\,\sqrt[3]{4}}=\dfrac{\sqrt{6}\,\sqrt[3]{4}}{2}=$ etc.

17. $\sqrt[4]{8}\div\sqrt{2}$.

18. $\sqrt{32}\div\sqrt[4]{2}$.

19. $\sqrt[3]{\tfrac{1}{4}}\div\sqrt{\tfrac{1}{2}}$.

20. $3\div(2-\sqrt{3})$.

HINT. $3\div(2-\sqrt{3})=\dfrac{3}{2-\sqrt{3}}=\dfrac{3\,(2+\sqrt{3})}{(2-\sqrt{3})\,(2+\sqrt{3})}=$ etc.

21. $\sqrt{3}\div(\sqrt{2}+\sqrt{3})$.

22. $\sqrt{7}\div\sqrt{2-\sqrt{3}}$.

23. $\sqrt{2-\sqrt{3}}\div\sqrt{3-\sqrt{2}}$.

24. $(\sqrt{7}+\sqrt{5})\div(2\sqrt{7}-\sqrt{5})\div(19-3\sqrt{35})$.

25. Find to four decimals the numerical value of the results in Exercises (*a*) 20, (*b*) 21, and (*c*) 22.

26. In Exercise 21 divide the numerical value of the numerator by the numerical value of the denominator, each having been obtained to five decimals. Compare the quotient with the result obtained for that fraction in Exercise 25.

27. What conclusion can be drawn from Exercises 25 and 26 regarding the rationalization of the denominator of a fraction before finding its numerical value?

Change to respectively equivalent fractions having rational denominators:

28. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$. **30.** $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$. **32.** $\dfrac{\sqrt{x-3}+\sqrt{3}}{\sqrt{x-3}-\sqrt{3}}$.

29. $\dfrac{2\sqrt{5}+3\sqrt{7}}{3\sqrt{5}-2\sqrt{7}}$. **31.** $\dfrac{\sqrt{x}-2\sqrt{c}}{\sqrt{x}+\sqrt{c}}$. **33.** $\dfrac{4}{\sqrt[4]{2-\sqrt{2}}}$.

Perform the indicated division:

34. $(\sqrt{10}-\sqrt{5})\div(\sqrt{10}+\sqrt{5})$. **35.** $(x-\sqrt{c})\div(x-3\sqrt{c})$.

36. $(\sqrt{a+c}-\sqrt{x})\div(\sqrt{a+c}+\sqrt{x})$.

37. $(\sqrt{3}+\sqrt{2})\div(2-\sqrt{3}+\sqrt{2})$.

38. $(\sqrt{5}-\sqrt{7})\div(\sqrt{5}+\sqrt{7}-\sqrt{2})$.

39. Is there any real distinction between the direction before Exercise 28 and that before Exercise 34?

40. Does $3-\sqrt{7}$ satisfy $x^2-6x+2=0$?

41. Does $\dfrac{7-\sqrt{3}}{2}$ satisfy $2x^3-75x+161=0$?

42. Does $\frac{1}{6}(5\pm\sqrt{109})$ satisfy $3x^2-5x-7=0$?

141. Square root of surd expressions. The square of a binomial is usually a trinomial. However, the result of squaring a binomial of the form $\sqrt{a}+\sqrt{b}$ is a binomial, if a and b are rational numbers. Thus $(\sqrt{7}-\sqrt{3})^2=7-2\sqrt{21}+3=10-2\sqrt{21}$. Here in $10-2\sqrt{21}$, 10 is the sum of 7 and 3, and 21 is the product of 7 and 3. These relations, and the fact that the coefficient of the radical $\sqrt{21}$ is 2, enable us to find the square root of many expressions of the form $a\pm2\sqrt{b}$ by writing each in the form of $x\pm2\sqrt{xy}+y$ and then taking the square root of the trinomial square as follows:

Example: Extract the square roots of $9-\sqrt{56}$.

Solution: $9-\sqrt{56}=9-2\sqrt{14}$.

We now find two numbers whose sum is 9 and whose product is 14. These are 7 and 2.

Therefore $9-2\sqrt{14}=2-2\sqrt{14}+7=(\sqrt{2}-\sqrt{7})^2$.

Hence the square roots of $9-\sqrt{56}$ are $\pm(\sqrt{2}-\sqrt{7})$.

EXERCISES

Find the positive square roots in Exercises 1–12:

1. $6 - 2\sqrt{8}$.

2. $7 + 2\sqrt{10}$.

3. $13 + \sqrt{48}$.

4. $8 - \sqrt{60}$.

5. $11 - 4\sqrt{7}$.

6. $17 + 12\sqrt{2}$.

7. $11 - 3\sqrt{8}$.

8. $65x - 20\sqrt{3x^2}$.

9. $126a - 10a\sqrt{5}$.

10. $\frac{13}{4}a - \sqrt{3a^2}$.

11. $2x + 2\sqrt{x^2 - 49}$.

12. $a + \sqrt{a^2 - 1}$.

13. $\sqrt{9 + 3\sqrt{8}} = \sqrt{?} + \sqrt{?}$

14. $\sqrt{15 - 5\sqrt{8}} = ?$

15. $\sqrt{m^2 + m + 2n + 2m\sqrt{m + 2n}} = ?$

Note. In the writings of one of the later Hindu mathematicians (about 1150 A.D.) we find a method of extracting the square root of surds, which is practically the same as that given in the text. In fact, the formula for the operation is given, apart from the modern symbols, as follows: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$. The study of expressions of the type $\sqrt{\sqrt{a} \pm \sqrt{b}}$ had been carried to a most remarkable degree of accuracy by the Greek, Euclid. His researches on this subject, if original with him, place him among the keenest mathematicians of all time; but his work and all of his results are expressed in geometrical language, which is very far removed from our algebraic symbolism, and for that reason is little read now.

142. Factors involving radicals. Review § 96, pages 256–257.

EXERCISES

Factor:

1. $x^2 - 11$.

2. $3x^2 - 16$.

3. $x^3 + 2$.

4. $x^3 - 12$.

5. $3x^3 - 27$.

6. $5x^3 + 125$.

Find the algebraic sum of:

7. $\frac{2\sqrt{b}}{a - b} + \frac{2}{\sqrt{a} + \sqrt{b}}$.

8. $\frac{x + c}{\sqrt{x} - \sqrt{c}} - \frac{x^{\frac{3}{2}} + c^{\frac{3}{2}}}{x - c}$.

Solve by factoring and check:

9. $x^2 - 5 = 0$.

10. $2x^2 - 3 = 0$.

11. $x^4 + 144 = 26x^2$.

12. $4x^4 + c = x^2 + 4cx^2$.

PROBLEMS

(Obtain answers in simplest radical form.)

1. The side of an equilateral triangle is 12; find the altitude.

2. The side of an equilateral triangle is s; find the altitude and the area.

3. The altitude of an equilateral triangle is 20; find one side and the area.

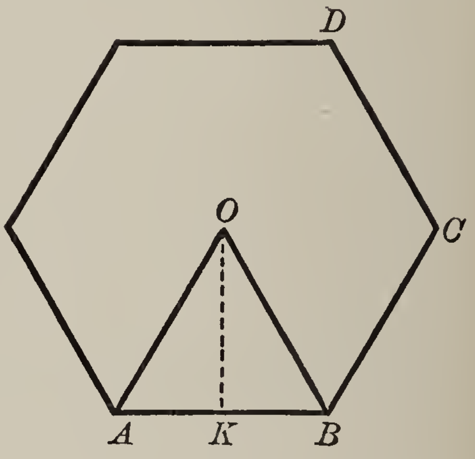
4. Find the side of an equilateral triangle whose altitude is a.

5. Find the altitude on the shortest side of the triangle whose sides are 9, 10, and 17. Find the area of the triangle.

6. Find the altitude on the longest side of the triangle whose sides are 10, 12, and 16.

Fact from Geometry. A regular hexagon may be divided into six equal equilateral triangles by lines from its center to its vertices.

In the adjacent regular hexagon $AB = BC = CD$, etc. O is the center and OK is the *apothem* of the hexagon.



7. Find the apothem and the area of a regular hexagon (a) whose side is 15; (b) whose side is s.

8. Find the side and the area of a regular hexagon (a) whose apothem is 25; (b) whose apothem is h.

Fact from Geometry. The volume of a pyramid or cone is $\frac{a \cdot b}{3}$, where a is the altitude and b is the area of the base.

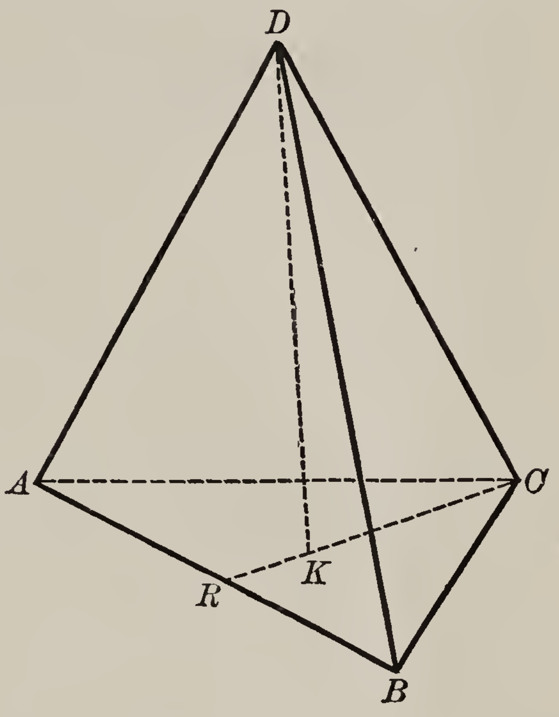
9. The base of a pyramid is a square, each side of which is 10 feet. The other four edges are each 20 feet. Find the altitude and the volume of the pyramid.

10. The base of a pyramid is a rectangle 8 by 18. The other four edges are each 16. Find the altitude of the pyramid.

11. The side of an equilateral triangle is 18. Find the two parts into which each altitude is divided by the other altitudes.

Fact from Geometry. The altitudes of an equilateral triangle intersect at a point which divides each altitude into two parts whose ratio is 2 to 1.

The altitude of a regular tetrahedron (*DK* in the adjacent figure) meets the base at the point where the altitudes of the base intersect.



12. *ABCD* is a regular tetrahedron. If each edge is 12, find *CR*, *CK*, and lastly the altitude *DK*.

Fact from Geometry. A regular tetrahedron is a pyramid whose four sides are equal equilateral triangles.

13. Find the altitude and volume of a regular tetrahedron whose edge is 15.

14. Show that the altitude and the volume of a regular tetrahedron whose edge is e are respectively $\frac{e}{3}\sqrt{6}$ and $\frac{e^3}{12}\sqrt{2}$.

15. The base of a pyramid is a regular hexagon each side of which is 10 inches. The other edges of the pyramid are each 16 inches. Find the altitude and the volume of the pyramid.

CHAPTER XXIX

EXPONENTS

143. Fundamental laws of exponents. The laws of exponents may be stated as follows:

I. Law of Multiplication,

$$x^a \cdot x^b = x^{a+b}.$$

Law I may be stated more completely thus:

$$x^a \cdot x^b \cdot x^c \cdots = x^{a+b+c}.$$

This follows directly from the definition of an exponent and from the Associative Law. For instance, $xx = x^2$, and $xxx = x^3$, and $xxxxx = x^5$ by definition. Hence $xx \cdot xxx = xxxxx$, or $x^2 \cdot x^3 = x^5$.

II. Law of Division,

$$x^a \div x^b = x^{a-b}.$$

This follows from Law I. For by that law $x^a = x^{a-b} \cdot x^b$. Hence, dividing both sides of the equation by x^b, we have $x^a \div x^b = x^{a-b}$.

III. Law of Involution, or raising to a power,

$$(x^a)^b = x^{ab}.$$

This follows from Law I, when instead of the distinct factors x^a, x^b, and x^c we have b factors, each equal to x^a.

IV. Law of Evolution, or the definition of a fractional exponent,

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}.$$

Law III includes the more general forms

(1) $$(x^a y^b)^c = x^{ac} y^{bc}.$$

(2) $$(((x^a)^b)^c) \cdots = x^{abc}.$$

It will be assumed that these laws hold for all real values of a, b, and c, excepting under IV, where b cannot be zero.

From § 106 it follows that: *Any* **factor** *of the numerator of a fraction may be omitted from the numerator and written as a* **factor** *of the denominator, and vice versa, if the sign of the exponent of the factor be changed.*

Therefore any expression involving negative exponents may be written as an expression involving only positive exponents. That is to say, negative exponents are not a mathematical necessity, but merely a convenience. The extension of the laws of exponents which brings with it the zero and the negative exponent is another illustration of the Law of Permanence of Form mentioned on page 61.

EXERCISES

Write with positive exponents and then simplify results:

1. 3^{-2}.

2. 4^{-3}.

3. $2^{-4} \cdot 3^0$.

4. $2^{-2} \cdot 3^{-4}$.

5. $7 \cdot 7^0 \cdot 0$.

6. $(\frac{1}{2})^{-3}$.

7. $(\frac{2}{3})^{-2} \cdot 4^0$.

8. $(\frac{4}{5})^{-3} \cdot (\frac{10}{3})^{-2}$.

9. $\dfrac{2}{3^{-2}}$.

10. $\dfrac{3}{3^0}$.

11. $\dfrac{12}{4^{-1}}$.

12. $5 \cdot 2^0 - (5 \cdot 2)^0$.

13. $\dfrac{4^{-2} \cdot 3^{-2}}{6^{-2}}$.

14. $(m - n)^0$, if $m \neq n$.

15. $32^{-\frac{2}{5}}$.

16. $0^5 \cdot 5^0$.

17. $4^{-\frac{1}{2}}$.

18. $8^{-\frac{1}{3}}$.

19. $16^{-\frac{1}{4}}$.

20. $8^{-\frac{2}{3}}$.

21. $16^{-\frac{3}{2}}$.

22. $25^{1.5}$.

23. $0^3 \cdot 0^{\frac{3}{4}}$.

24. $(-8)^{-\frac{2}{3}}$.

25. $(-64)^{-\frac{2}{3}}$.

26. $(-32)^{\frac{1}{5}}$.

27. $(32)^{-.4}$.

28. $(-125)^{-\frac{2}{3}}$.

29. $\sqrt[3]{27^{-2}}$.

30. $\sqrt[3]{8^{-2}}$.

31. $(\sqrt[3]{-8})^2$.

32. $(\frac{1}{2})^{-4} \cdot (\frac{1}{3})^{-3} \cdot (\frac{1}{2})^0$.

33. $(\frac{1}{6})^{-2}$.

34. $(.04)^{\frac{3}{2}}$.

35. $(.027)^{-\frac{2}{3}}$.

36. $(.064)^{-\frac{1}{3}}$.

37. $(.00032)^{\frac{2}{5}}$.

38. $\dfrac{\sqrt[3]{9^{-3}} \cdot \sqrt[2]{9^{-2}}}{3^{-4}}$.

39. $\dfrac{2^{-1}}{2^{-2} - 2^{-3}}$. HINT. $\dfrac{2^{-1}}{2^{-2} - 2^{-3}} = \dfrac{\frac{1}{2}}{\frac{1}{2^2} - \frac{1}{2^3}}$, etc.

40. $\dfrac{3^{-2} - 2^{-2}}{3^{-1} - 2^{-1}}$.

41. $\dfrac{2^{-1} + 3^{-1}}{2^{-3} + 3^{-3}}$.

42. $\dfrac{3^{-3} - 2^{-3}}{3^{-1} - 2^{-1}}$.

In this chapter it must be remembered that the letters in an expression may not take on such values as would make any denominator zero or any expression zero to the zero power.

Write with positive exponents and simplify results:

43. m^{-3}.

44. $2a^{-3}$.

45. $3ab^{-2}$.

46. $7x^2y^{-2}$.

47. $x^{-1}y^{-2}z$.

48. $4a^3b^{-2}c^2$.

49. $\dfrac{3}{a^{-2}}$.

50. $\dfrac{4x}{y^{-3}}$.

51. $\dfrac{4c^0}{xy^{-2}}$.

52. $\dfrac{4a^{-2}b^0}{y^{-5}}$.

53. $\dfrac{5^{-1}(ab)^0}{10^{-2}b^2}$.

54. $\dfrac{3a^3b^{-2}c}{4a^{-2}b^0}$.

55. $\dfrac{12x^2y^{-1}}{2yx^{-1}}$.

56. $\dfrac{10^{-1}a}{bc^3}$.

57. $\dfrac{4a^{-2}bc^{-3}}{6a^{-2}b^{-3}c^0}$.

58. $\dfrac{4^{-3}r^{-6}s^6}{s^{-2}r^{-2}t^3}$.

59. $\dfrac{2s^{-1}}{m^na^{-b}}$.

60. $\dfrac{5x^2y^a}{x^{-m}y^{-4}}$.

61. $\dfrac{2}{a^{-2}-b^{-2}}$. HINT. $\dfrac{2}{a^{-2}-b^{-2}}=\dfrac{2}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}$, etc.

62. $\dfrac{3}{a^{-1}+b^{-1}}$.

63. $\dfrac{a}{a^{-2}-b^{-2}}$.

64. $\dfrac{5se^2}{s^{-2}+e^{-2}}$.

65. $\dfrac{a^{-2}}{a^{-2}+b^{-2}}$.

66. $\dfrac{a^{-3}b^{-3}}{a^{-3}+b^{-3}}$.

67. $\dfrac{a^{-2}-b^{-2}}{a^{-1}+b^{-1}}$.

68. $\dfrac{a^{-4}-b^{-4}}{a^{-2}-b^{-2}}$.

69. $\dfrac{a^{-1}+b^{-1}}{a^{-3}+b^{-3}}$.

70. $\dfrac{a^{-3}-27^{-1}}{a^{-1}-3^{-1}}$.

Write without a denominator:

71. $\dfrac{2xy}{z^2}$.

72. $\dfrac{4a^3}{b^3}$.

73. $\dfrac{3x}{a^{-2}b^4}$.

74. $\dfrac{4sc^{-3}}{2^{-1}s^{-2}}$.

75. $\dfrac{12a^2b^3}{4xy^2}$.

76. $\dfrac{7x^{-1}y^2}{2^{-1}y^3}$.

77. $\dfrac{5ac^{-2}}{c(x-y)^2}$.

78. $\dfrac{7m^{-3}n^{\frac{1}{2}}}{m^{\frac{2}{3}}(m-n)^0}$.

79. $\dfrac{1}{5a^2(c+d)^{-3}}$.

80. $\dfrac{a(x-y)^{-2}}{bcx(x-y)}$.

81. $\dfrac{42m^{-n}n^{2m}}{56m^{-2n}n^{-3n}}$.

82. $\dfrac{r^{-1}s^2}{r^{-2}s^3(s-r)^3}$.

EXERCISES IN MULTIPLICATION

(Exercises 1–32 are oral.)

Perform the indicated multiplications:

1. $x^4 \cdot x^{-2}$.
2. $x^{\frac{1}{3}} \cdot x^{\frac{1}{2}}$.
3. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} \cdot x^0$.
4. $7\,x^0 \cdot x^{\frac{2}{3}} \cdot x^{\frac{3}{2}}$.
5. $\sqrt[3]{x} \cdot x^{\frac{1}{4}}$.
6. $\sqrt[3]{x} \cdot \sqrt[4]{x^3}$.
7. $a\sqrt{ax} \cdot a^{\frac{1}{2}}\sqrt{ax^{-1}}$.
8. $e^x \cdot e^{-x}$.
9. $e^{a-3} \cdot e^3$.
10. $e^{3-2a} \cdot e^{2+3a}$.
11. $x \cdot x^a \cdot x^b \cdot x^0 \cdot x^{2b-3a}$.
12. $(2^3)^3$.
13. $(2^{-3})^{-2}$.
14. $(x^2)^3$.
15. $(x^{\frac{1}{3}})^{\frac{1}{2}}$.
16. $(3\,x^{-2})^3$.
17. $(25\,a^4b^6)^{-\frac{1}{2}}$.
18. $(a^2b)^3(a+b)$.
19. $(a^3)^{2x} \cdot (a^2)^{3x}$.
20. $(a^3)^{x+y} \cdot (a^2)^{y-x}$.
21. $(x^2 - x^{-2})x^3$.
22. $(x^4 - a^4)x^{-2}a^{-2}$.
23. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})x^{\frac{2}{3}}y^{\frac{1}{3}}$.
24. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$.
25. $(a^{-2} + 3)(a^{-2} - 5)$.

Expand:

26. $(a^{-1} - a)^2$.
27. $(a^3 - 2\,a^{-2})^3$.
28. $(a^{-1} - 2\,a + 3\,a^{-2})^2$.
29. $(e^x + e^{-x})^2$.
30. $(a^{-1} + b^{-2})\left(\frac{1}{a} - \frac{1}{b^2}\right)$.
31. $(e^{2x} - 2 + e^{-2x})^2$.
32. $(3\,a^{-\frac{1}{2}} + 2\,a^{\frac{2}{3}})^2$.
33. $(a^{\frac{4}{5}} - 2\,a^{\frac{2}{5}}x + 4\,x^2)(a^{\frac{2}{5}} + 2\,x)$.
34. $(x^{\frac{1}{3}} + 2\,y^{\frac{1}{3}})(x^{\frac{2}{3}} - 2\,x^{\frac{1}{3}}y^{\frac{1}{3}} + 4\,y^{\frac{2}{3}})$.
35. $(a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}})$.
36. $(x - x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})(x + x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$.
37. $(\sqrt[3]{a^5} - 3\sqrt[2]{a})(\sqrt[3]{a^5} - 3\sqrt{a})$.
38. $(5\sqrt{c^{-5}}\;\sqrt[3]{d^{-3}} - a\sqrt[5]{c^{-2}})^3$.
39. $(\sqrt{a} + c^{-1}\sqrt{b})(\sqrt{a} + c^{-1}\sqrt{b})^2$.
40. $(m^{n-1} - 5\,m^{n-2}a^n + 25\,a^{2n})(m + 5\,a^n)$.
41. $(25\,x^{-4} + 15\,x^{-2}y^{-8} + 9\,y^{-16})(5\,x^{-2} - 3\,y^{-8})$.

Biographical Note. JOHN WALLIS. To us, who use the notation of exponents every day, it seems so simple and natural a method of expressing the product of several equal factors, that it is difficult to understand why such a long time was necessary to develop it. But here, as in many other instances, it required a great man to discover what to us seems the most obvious relation. The man who brought the notation of exponents to its modern form was John Wallis (1616–1703), an Englishman. He was the son of a clergyman, and, like most scholars of his day, did not confine his interests to any one subject. Wallis became widely known by deciphering a military dispatch which contained a hidden meaning, and all his life was interested in such puzzling problems. He was at one time an instructor in Latin, Greek, and Hebrew, wrote books on theology and English grammar, and invented a method of teaching deaf mutes to talk. He was the most notable English mathematician before Sir Isaac Newton, who highly prized Wallis's work.

Though the idea of using negative and fractional exponents had occurred to writers before Wallis, it was he who showed their naturalness, and who introduced them permanently. He also was the first to use the ordinary sign ∞ to denote infinity. His famous treatise on Algebra is noteworthy for its systematic use of symbols and formulas, and the insistence that letters of algebra are merely generalized numbers.

EXERCISES IN DIVISION

Perform the indicated division:

1. $x^4 \div x^6$.

2. $x^3 \div x^{\frac{1}{3}}$.

3. $x^{\frac{1}{2}} \div x^2$.

4. $ax^{\frac{2}{3}} \div x^{\frac{1}{2}}x^{\frac{1}{2}}$.

5. $\dfrac{ax - a^2x^2}{a^2x^{\frac{1}{2}}}$.

6. $(x^a - 2\,x^{2a-1} + 3\,x^{3a-2}) \div x^{2a-1}$.

7. $(6\,a^{3}$ [illegible] $- 9\,a^{n-2} + 5\,a^{2-n}) \div 3\,a^{n-2}$.

8. $(x - y) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

9. $(x - 8\,y) \div (x^{\frac{1}{3}} - 2\,y^{\frac{1}{3}})$.

10. $(16\,x^2 - 81\,y^2) \div (2\,x^{\frac{1}{2}} + 3\,y^{\frac{1}{2}})$.

11. $(a^3 - b^2) \div (\sqrt{a} + \sqrt[3]{b})$.

12. $(a^2 + ab^{-1} + b^{-2}) \div (a - a^{\frac{1}{2}}b^{-\frac{1}{2}} + b^{-1})$.

13. $(e^{2x} - 2 + e^{-2x}) \div (e^x - e^{-x})$.

14. $\left(e^{3x} + 3\,e^x + \dfrac{3}{e^x} + \dfrac{1}{e^{3x}}\right) \div (e^x + e^{-x})$.

15. $(a + a^{\frac{3}{5}}b^{\frac{2}{3}} + 2\,a^{\frac{2}{5}}b^{\frac{1}{3}} + 2\,b) \div (a^{\frac{3}{5}} + 2\,b^{\frac{1}{3}})$.

IOHANNES WALLIS S.T.P. GEOMETRIÆ PROF. SAVILIANUS OXON. REG. MA.tis A SACRIS REGALIS SOCIET. LOND. SODALIS
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JOHN WALLIS

Express in simplest form with positive exponents:

31. $(\sqrt[3]{-27x^9})^{-2}$.

32. $(\sqrt[4]{16a^4x^8})^{-2}$.

33. $\sqrt[2]{16^{\frac{3}{4}}x^9}$.

34. $(x\sqrt[3]{x^{-2}})^3$.

35. $(x^{\frac{3}{2}}\sqrt{x^{-4}})^{-\frac{2}{3}}$.

36. $(x^{-2}\sqrt{x^3\sqrt{4}})^4$.

37. $\left(x^{-2}\sqrt{x^3\sqrt[3]{x^2}}\right)^{-\frac{1}{2}}$.

38. $[(\sqrt[3]{8x^4})^{-2}]^{\frac{4}{3}}$.

39. $(\sqrt[2]{\sqrt[4]{16a^6}})^4$.

40. $\left[(27x^3)^{-\frac{1}{3}}\cdot\frac{1}{5x^{-4}}\right]^{-3}$.

41. $\left(a^{\frac{1}{2}}x^{-\frac{1}{3}}\sqrt{ux^{-\frac{1}{4}}\sqrt[3]{x^{\frac{5}{3}}}}\right)^{\frac{1}{2}}$.

42. $\sqrt[2]{\sqrt[2]{a}}$.

43. $\sqrt[3]{\sqrt[2]{x^4}}$.

44. $\sqrt[3]{\sqrt[4]{x^5}}$.

45. $\sqrt[2]{2a^4\sqrt[5]{x^3}}$.

46. $\dfrac{[\sqrt{(25x^2)^2}]^{-\frac{1}{3}}}{\sqrt[3]{5x}}$.

47. $(32)^{.4}$.

48. $16^{-1.5}$.

49. $4\cdot 2^5\cdot 2^{n-5}$.

50. $(\frac{2}{3})^{-2}\cdot(\frac{3}{2})^0\cdot 2^2-\dfrac{2(\frac{1}{8})^{\frac{1}{3}}}{8^{-\frac{2}{3}}}$.

51. $5\cdot 2^0-1^5+(5\cdot 2)^0$.

52. $\dfrac{4^{-3}-2^{-3}}{4^{-1}+2^{-1}}$.

53. $\dfrac{3^e\cdot 3-3^{4+e}}{9\cdot 3^{2+e}}$.

54. $\dfrac{a^{-1}+2x^{-1}}{a^{-3}+8x^{-3}}$.

55. $\dfrac{x^{-4}+x^{-2}+1}{x^{-2}-x^{-1}+1}$.

Simplify:

56. $(a^{n-1})\cdot(a^{3-n})^3\cdot(a^{2n+5})^{-1}$.

57. $a^{\frac{n+2}{n+1}}\div a^{\frac{1}{n+1}}$.

58. $a^{\frac{n}{n-1}}\div a^{\frac{n}{n+1}}$.

59. $a^{n^2-m^2}\div a^{n-m}$.

60. $\left((a^{n+1})^{\frac{1}{n^2-1}}\right)^{n^2-n}$.

61. $(\sqrt[n]{x^{n^2}}\div\sqrt[3]{x})^{\frac{3}{3n-1}}$.

62. $\dfrac{a^2\cdot 16^{\frac{1}{2}}a^{-\frac{3}{4}}}{a^{-\frac{7}{4}}+\dfrac{4}{a^{\frac{3}{4}}}}$.

63. $[(x-y)^2]^0$.

64. $\left(x^{\frac{1}{n^2}-1}\right)^{\frac{1}{n+1}}$.

65. $[32^{-\frac{3}{5}}\cdot 9^{\frac{3}{2}}]^{-\frac{1}{3}}$.

66. $\left(\dfrac{8x^3}{125y^{-3}}\right)^{-\frac{1}{3}}\cdot\left(\dfrac{m^{-3}n^2}{x^3y^{-2}}\right)^{-2}\cdot\left(\dfrac{2^4x^{-6}}{x^2y^8}\right)^{\frac{1}{2}}\cdot\left(\dfrac{2y^3}{m^2}\right)^{-3}\cdot\left(\dfrac{n^6}{x^3}\right)^{\frac{1}{3}}$.

67. $\dfrac{4^{n+1}}{2^n(4^{n-1})^n}\div\dfrac{8^{n+1}}{(4^{n+1})^{n-1}}+3+9x^0-(9x)^0-1^{9x}+9^0$.

68. $[((x^{-m})^n)^{-p}]\div[((x^{-n})^m)^p]$. **69.** $[(2x^2)^0\cdot(2x)^{\frac{1}{2}}\cdot(2x)^{-\frac{1}{2}}]^{-\frac{1}{5}}$.

70. $[(5a)^{2n}\cdot(5a)^{p-3n}]^{\frac{2}{3}}.$

71. $(a^{\frac{2}{3}}\cdot a^{\frac{5}{11}})^{\frac{33}{5}}.$

72. $\left(r^{-\frac{1}{2}}s^{\frac{2}{3}}\sqrt{r^3s^{-\frac{4}{3}}}\right)^{\frac{2}{5}}.$

73. $\sqrt[5]{x^{10}}+3(x^{-6})^{-\frac{1}{2}}.$

74. $\dfrac{3a^{-b}c^{-3}}{7x^{-4}y^{-d}}\cdot\dfrac{21x^{-2}y^{5-d}}{6a^bc^{d-3}}.$

75. $\dfrac{2^{2n}\cdot 3^{n+1}\cdot 6^{n+\frac{1}{2}}}{(2\cdot 3)^n\cdot 3^{n-\frac{1}{2}}}\div\dfrac{2^{\frac{3}{2}-n}\cdot 12^{2n+2}}{18^{n+2}}.$

76. $\left(\dfrac{x^{-2}y}{x^{\frac{3}{2}}y^{-\frac{5}{2}}}\right)^{\frac{2}{7}}\div\left(\dfrac{xy^{-\frac{1}{2}}}{\sqrt{x}\,y^{-1}}\right)^{-2}.$

77. $\dfrac{m^7n}{r^{-r}s^{-1}}\cdot\dfrac{m^sr^{r+2}}{n^{-5}s^{-3}}.$

78. $\dfrac{ab^{\frac{1}{2}}c\sqrt{a}\,b\sqrt[3]{c}}{\sqrt[3]{a^2}b^{-\frac{1}{2}}a^{\frac{1}{3}}b^2c^{\frac{4}{3}}}.$

79. $\dfrac{xy^{\frac{1}{3}}z\sqrt{xy}\sqrt[5]{z}}{\sqrt[2]{x^3}\cdot y^{-1}zy^3\sqrt[5]{z^6}x^{-2}}.$

Find the square roots of:

80. $x^6+4x^3y^{\frac{2}{3}}+9-4x^{\frac{9}{2}}y^{\frac{1}{3}}+6x^3-12x^{\frac{3}{2}}y^{\frac{1}{3}}.$

81. $\dfrac{x^2}{y^2}+\dfrac{4y^2}{x^2}+4+\dfrac{3x^2}{y^2}+\dfrac{12y^2}{x^2}-20.$

Find the indicated roots of :

82. $[(e^x-e^{-x})^2+4]^{\frac{1}{2}}.$

83. $[(e^x+2^{-1}e^{-x})^2-2]^{\frac{1}{2}}.$

84. $(e^{2x}+e^{-2x}+2+4e^x-4e^{-x})^{\frac{1}{2}}.$

85. $(x^{-6}+17x^{-2}+16x^2-6x^{-4}-24)^{\frac{1}{2}}.$

86. $\left(\dfrac{1}{x}-\dfrac{4}{\sqrt[4]{x^3}}+6x^{-\frac{1}{2}}-4x^{-\frac{1}{4}}+1\right)^{\frac{1}{4}}.$

87. $\left(\dfrac{4a^{-8}b^8}{9}+\dfrac{9a^{-4}b^4}{16}+\dfrac{16}{25}+a^{-6}b^6-\dfrac{16a^{-4}b^4}{15}-\dfrac{6a^{-2}b^2}{5}\right)^{\frac{1}{2}}.$

Simplify :

88. $\dfrac{\dfrac{x^3\cdot a\cdot x^0-a\,3x^2}{(x^3)^2}}{\dfrac{a}{x^3}}.$

89. $\dfrac{\dfrac{x^4(3x^2)-(x^3+5)4x^3}{(x^4)^2}}{\dfrac{x^3+20}{x^4}}.$

90. $\dfrac{\dfrac{(x^2-1)ax^{a-1}-x^a\cdot 2x}{(x^2-1)^2}}{\dfrac{x^a}{x^2-1}}.$

91. $\dfrac{\dfrac{x^2(ax^{a-1})-(x^a+1)2x}{(x^2)^2}}{\dfrac{x^a+1}{x^2}}.$

92. $$\frac{\dfrac{x^{2a} n x^{n-1} - x^n \cdot 2\,a x^{2a-1}}{(x^{2a})^2}}{\dfrac{x^n}{x^{2a}}}.$$

93. $$\frac{\dfrac{x^{-5}(-2\,x^{-1}) - (x^2+3)(-5\,x^{-4})}{(x^{-5})^2}}{\dfrac{x^{-2}+3}{x^{-5}}}.$$

94. $$\frac{\dfrac{e^{-nx}(ne^{nx}) - (e^{nx}+1)(-ne^{-nx})}{(e^{-nx})^2}}{\dfrac{2\,e^{nx}+1}{e^{-nx}}}.$$

95. $$\frac{\dfrac{(\sqrt[3]{x}+1)\frac{1}{3}x^{-\frac{2}{3}} - \sqrt[3]{x}\left(\frac{1}{3}x^{-\frac{2}{3}}\right)}{\left(\sqrt[3]{x}+1\right)^2}}{\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}+1}}.$$

96. $$\frac{\dfrac{\sqrt{ax-x^2}\cdot b - bx(ax-x^2)^{-\frac{1}{2}}\cdot(a-2\,x)}{(\sqrt{ax-x^2})^2}}{bx \div \sqrt{ax-x^2}}.$$

97. $$\frac{\dfrac{\left(\sqrt{x^{2n}-1}\right)2\,nx^{2n-1} - x^{2n}(x^{2n}-1)^{-\frac{1}{2}}(nx^{n-1})}{\left(\sqrt{x^{2n}-1}\right)^2}}{x^{2n} \div \sqrt{x^{2n}-1}}.$$

98. $$\frac{\dfrac{\sqrt{x^2-10\,x}\cdot 5 - 5\,x\cdot\frac{1}{2}(x^2-10\,x)^{-\frac{1}{2}}(2\,x-10)}{\left(\sqrt{x^2-10\,x}\right)^2}}{5\,x \div (x^2-10\,x)^{\frac{1}{2}}}.$$

99. Show that $\dfrac{\sqrt{m-n}}{\sqrt{m}+\sqrt{n}} = \dfrac{\sqrt{m}}{\sqrt{m-n}} - \dfrac{\sqrt{n}}{\sqrt{m-n}}$.

100. Show that $\dfrac{\sqrt[a+1]{\sqrt[a+2]{x}}}{\sqrt[a]{\sqrt[a+1]{x}}} = \dfrac{\sqrt[a+1]{x^2}}{\sqrt[a]{x}\cdot\sqrt[a+2]{x}}$.

CHAPTER XXX

GRAPHICAL SOLUTION OF EQUATIONS IN ONE UNKNOWN

144. Graph of a function. A graph always shows a relation between (at least) two variables. The graph of a function of one variable is a curve showing the value of the function for any real value of the variable. This means that one axis must be the x-axis and the other the function axis, or F-axis. The method of constructing the graph of a function of x is the same for a linear, a quadratic (see pages 259–266), and a cubic function in one variable.

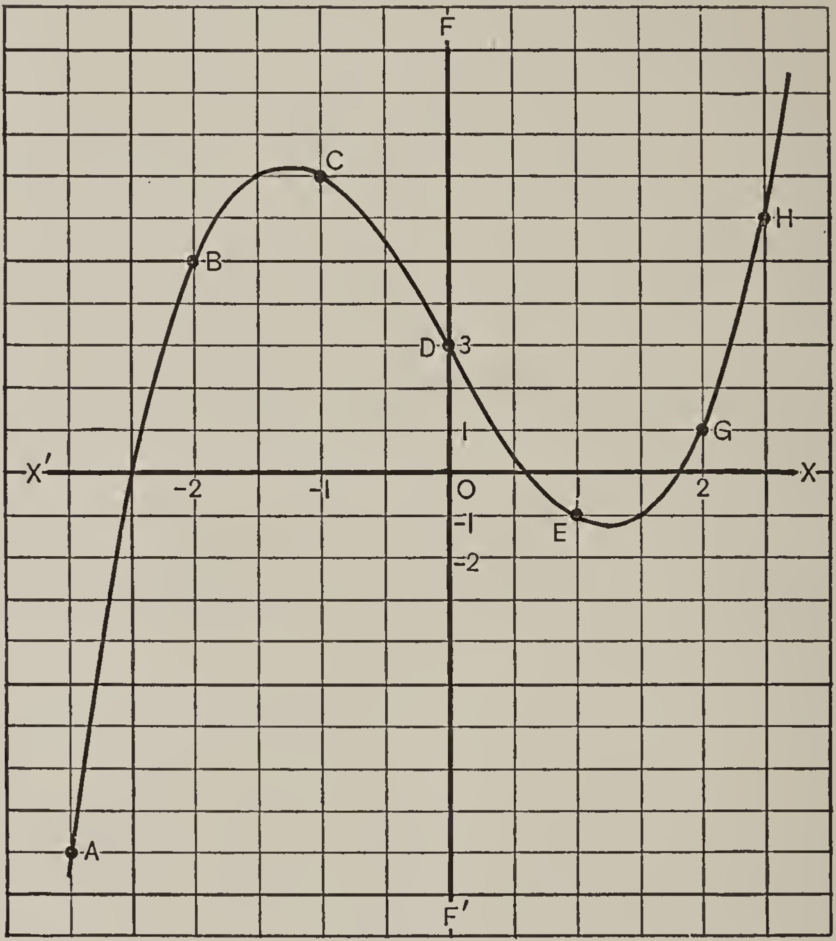
In physical science if one variable quantity depends on another, the first is said to be a function of the second. Thus the height of the mercury in a thermometer is a function of the temperature. Similarly, the space passed through by a falling body depends on (is a function of) the time of fall. This is expressed by the equation $s = 16\,t^2$. The variation of these two quantities is shown graphically on page 189.

One quantity may be a function of two others or of many others. Thus the time of one vibration of a pendulum is a function of two other quantities. The relation is expressed by the equation $T = \pi\sqrt{\frac{L}{g}}$, L being the length of the pendulum and g the measure of the earth's attractive force. Many other illustrations might be taken from astronomy and physics; in fact, the notion of a function underlies a vast number of problems with which the physical scientist has to deal.

145. Graph of a cubic function. To graph the function $x^3 - 5x + 3$, first prepare a table of values as follows:

When $x =$	-4	-3	-2	-1	0	1	2	$2\frac{1}{2}$	3
$f(x)$, $x^3 - 5x + 3 =$	-41	-9	5	7	3	-1	1	$6\frac{1}{8}$	15

Plotting the points corresponding to the numbers in the table (except the first and last), we obtain A $(-3, -9)$, B, C, D, E, G, and H, in the order named. The curve crosses the x-axis three times :



once between 1 and 2; again between 0 and 1; and a third time between -2 and -3. At the points of crossing $f(x)$ is zero. Therefore the values of x at these points are the roots of $x^3 - 5x + 3 = 0$. These are approximately 1.8, .6, and -2.5.

EXERCISES

(Exercises 12–16 refer to the preceding graph.)

1. Construct the graph of $f(x) = 3x - 9$.

2. Does the x-coördinate of the point where the line crosses the x-axis satisfy the equation $3x - 9 = 0$? Why?

3. What is the graph of any linear function of x?

4. Construct the graph of $f(x) = 2x^2 - x - 6$.

5. Do the x-coördinates of the points where the curve crosses the x-axis satisfy the equation $2x^2 - x - 6 = 0$? Why?

6. What kind of a line do you expect the graph of any quadratic function in one variable to be?

7. State a rule for the graphical solution of a linear or a quadratic equation in one variable.

8. What is the effect on the graph of a quadratic function of x, if a positive number is added to the constant term?

9. What change occurs in the roots of a quadratic equation in x, if a positive number is added to its constant term?

10. When does the graphical solution of a quadratic equation give but one real root?

11. When does the graphical solution of a quadratic equation fail to give the roots of the equation?

12. If the function $x^3 - 5x + 3$ be set equal to 4, can the roots of the equation thus formed be read from the graph? If so, read them.

13. Set $x^3 - 5x + 3$ equal to -1.3 (approximately) and read the roots of the resulting equation from the graph. Explain.

14. Set the function $x^3 - 5x + 3$ equal to -4 and read the roots of the resulting equation from the graph. Explain.

15. Set $x^3 - 5x + 3$ equal to 8 and read the roots of the resulting equation from the graph. Explain.

16. Set $f(x)$, $x^3 - 5x + 3$, equal to 9 and read the roots of the resulting equation from the graph. Explain.

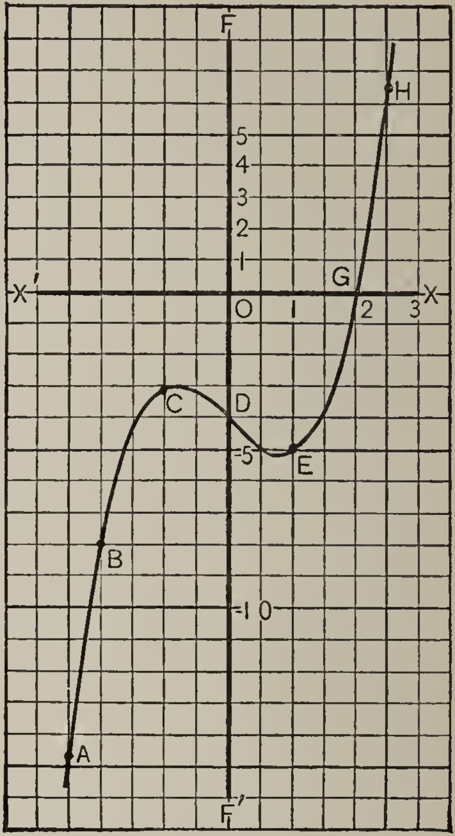
17. (*a*) Is a rational function always integral? (*b*) Is an integral function always rational? (*c*) Write an example of each.

146. Imaginary roots. To make clearer the point in Exercises 14–16 preceding, we shall graph the function $x^3 - 2x - 4$.

When $x =$	-3	$-2\frac{1}{2}$	-2	-1	0	1	2	$2\frac{1}{2}$	3
$f(x)$, $x^3 - 2x - 4 =$	-25	$-14\frac{5}{8}$	-8	-3	-4	-5	0	$6\frac{5}{8}$	17

The point A corresponds to $-2\frac{1}{2}$, $-14\frac{5}{8}$. The points corresponding to the next six pairs of numbers given are B, C, D, E, G, and H in the adjacent figure. The curve through these points crosses the x-axis but *once*. This shows that the equation has but one *real* root, and that the value of this root is 2. Since the number of roots of a rational integral equation is the same as the number which indicates its degree, we conclude that the other two roots are imaginary.

Note. It required the genius of Sir Isaac Newton first to observe from the graph of a function that two of its roots become imaginary simultaneously. He also saw that an equation with two of its roots equal to each other is, in a certain sense, the limiting case between equations in which the corresponding roots appear as two real and distinct roots, and those in which they appear as imaginary roots.



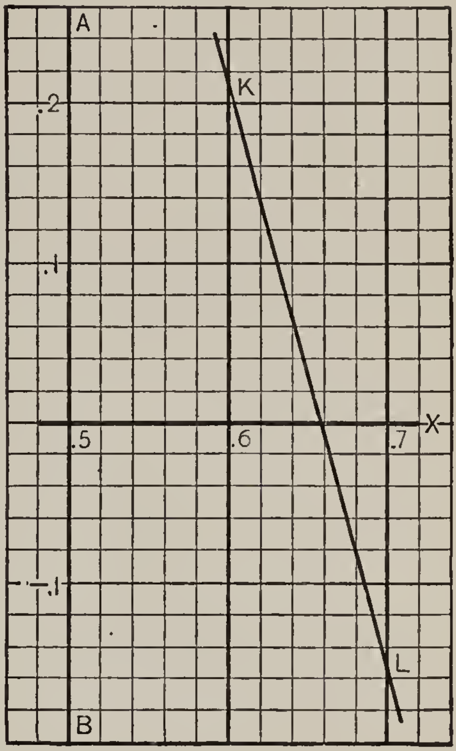
147. Graphical solution of an equation in one unknown. If the student has grasped the meaning of the preceding graphical work, he will see the correctness of the following rule for solving graphically *any* equation in one unknown.

RULE. *An equation in one unknown whose second member is zero is solved for* **real roots** *by graphing the function in the first member and then obtaining the value of x for the points where the curve crosses the x-axis.*

148. More accurate graphical solutions. By drawing the entire graph to a larger scale the student can obtain more accurately the values of the roots. If still more exact results are desired, he may proceed somewhat as follows:

The graph on page 362 shows that there is a root greater than .6 and less than .7. If we now construct on a large scale that portion of the curve between D and E (page 362) which is just above and just below the x-axis, we shall get a more precise value for the root. Substituting .6 and .7 in $x^3 - 5x + 3$, we obtain the following table:

When $x =$	.6	.7
$f(x)$, $x^3 - 5x + 3 =$	.216	$-.157$



Between $x = .6$ and $x = .7$ the function changes from + to −. Hence we are certain that the graph crosses the x-axis between these points. We now choose a much larger scale than the one used on page 362. This is indicated by the numbers on the x-axis. The scale is too large to show the y-axis in the figure, so the scale for y is indicated on the line AB. The point K corresponds to .6, .216, and the point L to .7, −.157. Since K and L are comparatively close together, the portion of the graph between them is nearly a straight line. Drawing the straight line KL, it is seen to cross the x-axis between .64 and .66, or about .658. By an algebraic method of solution it can be shown that the root, correct to three decimals, is .656. Here the graphical method gives the result to within $\frac{3}{10}$ of one per cent of the true value.

EXERCISES

Solve graphically:

(Obtain roots in Exercises 2 and 6 correct to two decimals.)

1. $x^2 + 14 = 8x$.

2. $x^3 - 3x + 4 = 0$.

3. $x^3 + x = 4$.

4. $x^3 - 2x^2 - 5x + 6 = 0$.

5. $x^3 - 8x = 0$.

6. $x^3 - 4x + 2 = 0$.

7. $x^4 - 10x^2 + 16 = 0$.

8. $x^4 - 4x^3 + 12 = 0$.

By reference to the curve obtained in Exercise 6 solve:

9. (*a*) $x^3 - 4x = -5$. (*b*) $x^3 - 10 = 4x$. (*c*) $x^3 - 4x - 2 = 0$.

149. Critical values of the variable. There are many practical problems involving two variables in which it is necessary to determine that value of one variable for which the other has the greatest (or least) possible value. A great number of these problems can be solved by means of a graph. The method of solution can be made clear by reference to the graph of § 145. There $f(x) = x^3 - 5x + 3$. Suppose we wish to know the value of x which gives $x^3 - 5x + 3$ the greatest possible value. Near C occurs a high point, — a turning point of the curve, — and there $x = -\frac{5}{4}$ approximately, and $f(x) = 7.3$. This value of x gives to $f(x)$ a greater value than does any other value of x between -2.5 and $+2.6$. It is true that on the portion GH above H greater values of $f(x)$ than 7.3 occur. But in practical problems similar to those of the next list it will be found that some condition of the problem will rule out of consideration any value of x which does not correspond to the turning point of the curve such as that which occurs near C or near E.

Biographical Note. The notion of a function is one of the most fundamental ideas in modern mathematics. Only the simplest examples are given in this book, but many others involving expressions of the utmost complexity have been studied by mathematicians for many years. An important reason for the study of functions is found in the fact that many kinds of facts and principles which we meet in the study of nature can be expressed symbolically by means of functions, and the discovery of the properties of such functions helps us to understand the meaning of the facts. A complete understanding of the laws of falling bodies, light, electricity, or sound could never be reached without the study of the mathematical functions which these phenomena suggest.

One of the foremost living scholars who has discovered many properties of the most complicated functions is Professor Felix Klein of Göttingen, Germany. Since the time of Gauss, who was also a professor at Göttingen, the university there has been one of the leading institutions of the world in the study of mathematics. It is interesting to know that Klein's great achievements in advanced mathematics have not caused him to forget the difficulties which surround the beginner in the first years of his study, but that he has had wide influence in improving mathematical instruction in the schools not only of Germany but of other countries as well.

PROBLEMS

1. A manufacturer has in stock a quantity of strawboard 8 inches by 15 inches, out of which he desires to make open-top boxes by cutting equal squares out of each corner and folding up so as to make sides and ends. What must the side of the square be so as to make a box of the greatest possible volume?

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HINT. Let x equal the side of the square in inches. Then the dimensions of the box in inches are $15 - 2x$, $8 - 2x$, and x. Hence the volume $V = 4x^3 - 46x^2 + 120x$ in cubic inches. Construct the graph of $V = 4x^3 - 46x^2 + 120x$ (or that of $V/4 = x^3 - 11\frac{1}{2}x^2 + 30x$, which deals with smaller numbers); then an inspection of the turning points will give the required value of x.

2. Referring to the graph of Exercise 1: (*a*) What value has the function $4x^3 - 46x^2 + 120x$ when $x = 1\frac{2}{3}$? (*b*) What other value of x gives the function the same value? (*c*) What values of x give the function greater values than this? (*d*) What condition of the problem rules out these values as sides of the square?

3. A piece of tin is 8 inches by 12 inches. From each corner a square whose side is x inches is cut out. The sides are then turned up and an open box is formed, which has the greatest possible volume. Find graphically this value of x.

4. What value of x gives $x^2 - 4x + 6$ the least possible value?

5. An open metal tank having a volume of 4 cubic yards has vertical sides and a square base. Determine the side of the base and the altitude of the tank if the inside surface is the least possible.

HINT. Let x equal the side of the base in yards and d the altitude in yards. Then the volume of the tank, 4 cubic yards, equals dx^2, and the surface equals $x^2 + 4dx$ in square yards. From these two statements we obtain surface $S = x^2 + \frac{16}{x}$. Plot the function $x^2 + \frac{16}{x}$ and the required value of x will be apparent.

6. An open metal tank having a volume of 4 cubic yards is in the form of a cylinder with a circular base. Determine the radius of the base and the altitude so that the inside surface will be the least possible.

7. The perimeter of a rectangle is 20 rods. Find the length and the width if the area is the greatest possible.

8. A boatman 6 miles from the nearest point of the beach (which is straight) wishes to reach in the shortest possible time a place 8 miles from that point along the shore. He can row 4 miles per hour and jog-trot 6 miles per hour. Determine where he must land.

Hint. Draw a right triangle ABC, AC being the shore line, B the boat, and A the point on shore nearest B. Let K on AC be the point at which he lands, and let KA in miles be x. Then $BK = \sqrt{x^2 + 36}$ and $CK = 8 - x$. In hours the time required to go from B to K is $\frac{\sqrt{x^2+36}}{4}$, and that from K to C is $\frac{8-x}{6}\cdot$ Therefore the total time equals $\frac{8-x}{6} + \frac{\sqrt{x^2+36}}{4}\cdot$ Plot this function and the required value of x will be apparent.

CHAPTER XXXI

QUADRATIC EQUATIONS

(*Review*)

150. Solution by completing the square. For the solution of the quadratic equation of the general type $ax^2 + bx + c = 0$ see Examples 1–3, pages 267–269, and the rule on page 269.

EXERCISES

Solve by completing the square and check.

(Find the values of the unknown in Exercises 4 and 5 correct to four decimals.)

1. $x^2 - 4x - 32 = 0.$

2. $2x^2 + 5x + 3 = 0.$

3. $\dfrac{x^2 + 3}{4} = \dfrac{5x^2 - 24}{7}.$

4. $x^2 - 5x + 2 = 0.$

5. $7x^2 - 12x - 3 = 0.$

6. $\dfrac{3x^2}{2} - 3x\sqrt{2} = 9.$

7. $(2x - 5)^2 - (x - 6)^2 = 80.$

8. $\dfrac{1}{x} + \dfrac{1}{x + 2} = \dfrac{3}{4}.$

9. $\dfrac{1}{w + 7} - \dfrac{1}{w - 3} = \dfrac{2}{5}.$

10. $\dfrac{m}{m + 1} - \dfrac{2m}{m + 2} + \dfrac{9}{2} = 0.$

11. $\dfrac{10s^2 - 1 - s^3}{6s - s^2 - 9} = \dfrac{s^2 - 9}{s + 3}.$

After a student has mastered the solution of quadratics by factoring and completing the square, he should learn the formula method (§ 102) and should use thereafter the one of the three methods which is best adapted to the problem in hand.

12. $\dfrac{4}{x - 4} - \dfrac{x + 3}{x^2 - 5x + 4} = 2.$

13. $\dfrac{2(x^3 + 8)}{x + 2} - \dfrac{x^3 - 1}{x - 1} = \dfrac{19}{4}.$

14. $\dfrac{r + 2\sqrt{5}}{\sqrt{15}} - \dfrac{2}{r} = \dfrac{1}{\sqrt{5}}.$

15. $x^3 + 7x^{\frac{3}{2}} - 8 = 0.$

The equation $x^3 + 7x^{\frac{3}{2}} - 8 = 0$ is not a quadratic, but it is of the general type $ax^{2n} + bx^n + c = 0$. Here x occurs in but two terms, and its exponent in one term is twice the exponent in the other. All equations of this form can be solved by completing the square.

Solution: $x^3 + 7x^{\frac{3}{2}} + \frac{49}{4} = 8 + \frac{49}{4} = \frac{81}{4}.$

$x^{\frac{3}{2}} + \frac{7}{2} = \pm\frac{9}{2}.$

$x^{\frac{3}{2}} = 1$ or -8; that is, $x^3 = 1$ or 64.

Whence $x = 1$ or 4.

Check: Substituting 1 for x in $x^3 + 7x^{\frac{3}{2}} - 8 = 0$,

$1 + 7 - 8 = 0$, or $0 = 0$.

Substituting 4 for x, $64 + 56 - 8 = 0$.

But $112 \neq 0$.

Hence the equation has only one real[*] root, 1.

16. $x^3 - 10x^{\frac{3}{2}} - 11 = 0.$

17. $x^4 - 26x^2 + 25 = 0.$

18. $x^6 - 7x^3 - 8 = 0.$

19. $x + x^{\frac{1}{2}} - 6 = 0.$

20. $4x^6 - 7x^3 = 15.$

21. $4x^2 + \frac{4}{x^2} - \frac{97}{9} = 0.$

22. $2x - 3x^{\frac{1}{2}} = 2.$

23. $3x^4 - 11x^2 + 6 = 0.$

24. $9x^4 - 22x^2 + 8 = 0.$

25. $3x^{\frac{3}{2}} + 5x^{\frac{3}{4}} + 2 = 0.$

26. $2x^{\frac{2}{3}} - 9\sqrt[3]{x} + 4 = 0.$

27. $3x - 11x^{\frac{1}{2}} - 20 = 0.$

28. $6x^4 - 13x^2 + 6 = 0.$

29. $x^{-2} + 16x^{-1} - 17 = 0.$

30. $y^{-4} - 10y^{-2} + 9 = 0.$

31. $x^{-1} - 13x^{-\frac{1}{2}} = -36.$

32. $x^{2m} + 4 - 5x^m = 0.$

33. $(x^2 - 2x)^2 - 7(x^2 - 2x) = -12.$

Solution: Let $x^2 - 2x = y.$

Substituting y for $x^2 - 2x$, we obtain

$y^2 - 7y = -12.$

Solving, $y = 3$ or 4.

Then $x^2 - 2x = 3.$

Whence $x = 3$ or -2.

Also $x^2 - 2x = 4.$

Whence $x = 1 \pm \sqrt{5}.$

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[*] The equations $x^3 = 1$ and $x^3 = 64$ have each two imaginary roots. Two of these roots satisfy $x^3 + 7x^{\frac{3}{2}} - 8 = 0$. It is not desirable to discuss these roots here, though it is well to point out their existence. The points involved are made clear in Exercises 7–10, page 473, in the chapter on Imaginaries.

In Exercises 34–39 do not expand or transpose and square. Solve as in Exercise 33.

34. $3(x^2+3x)^2-7(x^2+3x)-20=0.$

35. $\left(x-\frac{1}{x}\right)^2+4\left(x-\frac{1}{x}\right)=8\frac{1}{4}.$

36. $(4y+5)+2(4y+5)^{\frac{1}{2}}=15.$

37. $x^2+5x+3\sqrt{x^2+5x}-54=0.$

38. $x^2-2x-5\sqrt{x^2-2x-4}+2=0.$

39. $2y(2y+1)+3\sqrt{8y^2+12y+5}=25-4y.$

40. $2x^2+ax-6a^2=0.$

42. $7a^2x^2-4ax-11=0.$

41. $2x^3-17bx^2+8b^2x=0.$

43. $ax^2+bx+c=0.$

44. $12kx-4k^2-5x^2=0.$

45. $c^2(x-d)^2-d^2(c-x)^2=0.$

46. $mx^2-x(m^2+1)=-m.$

47. $(x-r)^2+(S-x)^2=r^2+S^2.$

48. $\frac{m}{m-x}-\frac{m-x}{m}=2.$

50. $\frac{x^2-3mx}{m-n}+2m=\frac{nx}{n-m}.$

49. $\frac{x-c}{c}-\frac{c}{x-c}=\frac{3}{2}.$

51. $\frac{x^2+2ax}{a-b}-\frac{x}{2b}=\frac{a}{b}.$

52. $\frac{2s+x}{x+s}-\frac{5s-x}{x-s}-\frac{s(x+s)}{s^2-x^2}=0.$

53. $2x^2+5x=cx^2+3cx+3.$

HINT. $x^2+\frac{5-3c}{2-c}x=\frac{3}{2-c}.$

$$x^2+\frac{5-3c}{2-c}x+\left(\frac{5-3c}{4-2c}\right)^2=\frac{3}{2-c}+\frac{25-30c+9c^2}{16-16c+4c^2}.$$

$$\left(x+\frac{5-3c}{4-2c}\right)^2=\frac{49-42c+9c^2}{16-16c+4c^2}=\left(\frac{7-3c}{4-2c}\right)^2.$$

$$x+\frac{5-3c}{4-2c}=\pm\frac{7-3c}{4-2c}, \text{ etc.}$$

54. $ax^2+3x^2+ax-5x-a+1=a-1.$

55. $cx^2-ax^2+cx+ax-c=c+a-ax.$

56. $x^2 + 2\,s + s^2 = 2\,sx + 2\,x.$

57. $x^2 - 2\,x + 1 = ax - ax^2.$

58. $x^2 + 2\,x + 1 = hx^2 + hx.$

59. $cx^2 + 3\,x = 2\,x^2 + 2\,cx - 2.$

60. $\dfrac{a}{x+a} + \dfrac{b}{x+b} - \dfrac{2\,c}{x+c} = 0.$

61. $\dfrac{a+x}{b+x} + \dfrac{b+x}{a+x} = \dfrac{5}{2}.$

62. $\dfrac{ax+b}{bx+a} = \dfrac{mx-n}{nx-m}.$

63. $\dfrac{ax^2+bx+c}{bx^2-mx+n} = \dfrac{c}{n}.$

151. Solution by formula. The standard form of the general quadratic is

$$ax^2 + bx + c = 0.$$

The student solved this equation (see Exercise 43, page 371) and found

$$x = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}. \qquad (F)$$

For the use of (F) see § 102, page 272.

EXERCISES

Solve for x by formula and check:

1. $2\,x^2 - 7\,x + 3 = 0.$

2. $3\,x^2 - x - 2 = 0.$

3. $11\,hx + 20\,h^2 = 3\,x^2.$

4. $5\,x^2 + 2\,cx = 16\,c^2.$

5. $22\,x^2 = 3\,mx + 7\,m^2.$

6. $6\,x^2 - 7\,rx + 2\,r^2 = 0.$

7. $3\,x^2 - 6\,ax + 2\,a^2 = 0.$

8. $3\,m^2 + 4\,mx - 7\,x^2 = 0.$

9. $4\,ax - 10\,a^2x^2 + 3 = 0.$

10. $12\,v^4 + 7\,v^2x - 10\,x^2 = 0.$

11. $x^2 + 3\,x = mx + 3\,m.$

HINT. $x^2 + (3 - m)x - 3\,m = 0.$ Then $a = 1, b = 3 - m,$ and $c = 3\,m.$ Substituting in (F), $x = \dfrac{-(3-m) \pm \sqrt{(3-m)^2 - 4 \cdot 1(-3\,m)}}{2}$, etc.

12. $x^2 + nx = cx + cn.$

13. $3\,x^2 - 6\,cx + 2\,c = x.$

14. $mx^2 + kmx = kc + cx.$

15. $a^2x^2 - 2\,ax = b^2x^2 - 1.$

16. $n^3x^2 + 2\,nx = 5\,n^2x + 10.$

17. $hkx^2 - hk = h^2x + h^2x^2.$

18. $cx^2 + cmx + 5 = cx + 5(x + m).$

19. $n^2x + 3\,nx + 2\,x = nx^2 + 2\,n + 3\,n^2.$

20. $m^2x^2 + 4\,mx + hmx + 3\,hx = 9\,x^2 + 12\,x - 4\,h.$

PROBLEMS

1. Separate 20 into two parts, such that the first shall be the square of the second.

2. One leg of a right triangle is 8 feet and the hypotenuse is 2 feet longer than the other leg. Find the other leg, the hypotenuse, and the area.

3. The number of hours required to make a trip of 112 miles was 6 more than the rate in miles per hour. Find the rate and the time.

4. The sum of the reciprocals of two consecutive numbers is $\frac{5}{6}$. Find the numbers.

5. The altitude of a triangle is 4 feet less than the base. The area of the triangle is 48 square feet. Find the base and the altitude.

6. One leg of a right triangle is 7 feet shorter than the other and the area is 30 square feet. Find the three sides of the triangle.

7. The area of a triangle is 40 square yards and the base is 2 feet more than seven times the altitude. Find the base and the altitude.

8. The area of a trapezoid is 60 square feet. One base is 2 feet more than the altitude and the other base is twice the altitude. Find the bases and the altitude.

9. A requires 4 more days than B to do a piece of work. If in working together they require $8\frac{8}{9}$ days, find the number of days each requires alone.

10. The radius of a circle is 21 inches. How much must it be shortened so as to decrease the area of the circle 770 square inches? (Use $\pi = \frac{22}{7}$.)

11. From a cask full of pure wine 5 gallons are drawn off. The cask is then filled by adding pure water, and again 5 gallons are drawn off. If 36 per cent of the mixture is water, how many gallons does the cask contain?

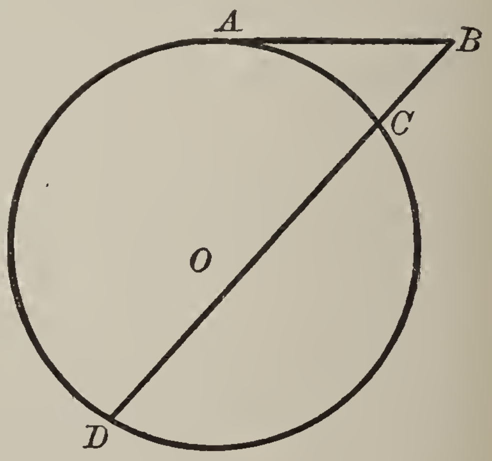
12. A printed page has 15 more lines than the average number of letters per line. If the number of lines is increased by 15, the number of letters per line must be decreased by 10 in order that the amount of matter on the two pages may be the same. How many letters are there on the page?

13. A sum of $5000 is put at interest. At the end of each year the yearly interest and $300 are added to the investment. If at the beginning of the third year the investment amounts to $6236, find the rate of interest that the investment bears.

14. Two bodies, A and B, move on the sides of a right triangle. A is now 123 feet from the vertex of the right angle and is moving away from it at the rate of 239 feet per second. B is 239 feet from the vertex and moves toward it at the rate of 123 feet per second. At what time (past or future) are they 850 feet apart?

15. The dimensions of a rectangular box in inches are expressed by three consecutive numbers. The surface of the box is 292 square inches. Find the dimensions.

16. A three-inch square is cut from each corner of a square piece of tin. The sides are then turned up and an open box is formed, the volume of which is 300 cubic inches. Find in inches the side of the piece of tin.



17. If AB in the adjacent figure is a tangent to the circle and BD is any secant, $\overline{AB}^2 = BC \cdot BD$. Find BC if $AB = 12$ and $CD = 20$.

If BC is small compared to DC, we may use $\overline{AB}^2 = BC \cdot CD$ as a close approximation. Thus if BC is a mountain two miles high and DC is the diameter of the earth, the equation $BC = \overline{AB}^2 \div CD$ would give the height of the mountain within one four-thousandth of the correct value.

18. How high is a mountain which can just be seen from a point on the surface of the sea 80 miles distant? (Use 3960 miles for the radius of the earth.)

19. Find the distance a man can see in a straight line over a smooth lake, if his eye is 6 feet above the level of the water.

20. Two lighthouses on opposite shores of a bay are 150 and 250 feet respectively above the water. If the light from one can just be seen from the other, find the distance in miles between them.

21. A stone dropped from a balloon which was passing over a river struck the water 12 seconds later. How high was the balloon at the time the stone was dropped?

HINT. The distance S through which a body falls from rest in t seconds is given by the equation $S = \frac{gt^2}{2}$ ($g = 32$ feet, approximately).

22. A man drops a stone over a cliff and hears it strike the ground below 13 seconds later, If sound travels 1120 feet per second, find the height of the cliff.

23. A messenger leaves the rear of an army 28 miles long as it begins its day's march. He goes to the front and at once returns, reaching the rear as the army camps for the night. How far did he travel if the army went 28 miles during the day?

CHAPTER XXXII

IRRATIONAL EQUATIONS

152. Definitions and typical solutions. An **irrational** or **radical equation** in one unknown is an equation in which the unknown letter occurs in a radicand.

Thus $3x + 2\sqrt{x} = 16$, $\sqrt{1-x} + \sqrt{x+3} = 2$, and $\sqrt[3]{x^2-8} = 0$ are irrational equations. Also any equation in which the unknown occurs with a fractional exponent is irrational.

The following examples illustrate the method of solution for some of the more simple irrational equations.

EXAMPLES

1. Solve $\sqrt{2x-5} - 3 = 0$.

Solution: Transposing, $\sqrt{2x-5} = 3$.
Squaring both members, $2x - 5 = 9$.
Solving, $x = 7$.

Check: Substituting 7 for x in the original equation,

$$\sqrt{14-5} - 3 = 0.$$

Whence $3 - 3 = 0$.

In irrational equations it is understood that each radical expression, not preceded by the sign ±, is to have *one* sign and *only one*; therefore each radical will have *one* value and *only* one. That value is the *principal root* of the radical. This fact is of importance in checking.

2. Solve $2\sqrt[3]{8x^3 + \dfrac{19x^2}{2}} - 2x - 2 = 2x - 1.$ (1)

Solution: Transposing, $2\sqrt[3]{8x^3 + \dfrac{19x^2}{2}} = 4x + 1.$ (2)

Cubing each member of (2),

$$64\,x^3 + 76\,x^2 = 64\,x^3 + 48\,x^2 + 12\,x + 1. \quad (3)$$

Transposing and collecting,

$$28\,x^2 - 12\,x - 1 = 0.$$

Factoring, $(2\,x - 1)(14\,x + 1) = 0.$

Therefore $x = \frac{1}{2}$ or $-\frac{1}{14}.$

Check: Substituting $\frac{1}{2}$ for x in (1),

$$2\sqrt[3]{1 + \tfrac{19}{8}} - 1 - 2 = 1 - 1.$$

$$2 \cdot \tfrac{3}{2} - 3 = 0.$$

$$3 - 3 = 0.$$

Substituting $-\frac{1}{14}$ for x in (1),

$$2\sqrt[3]{8(-\tfrac{1}{14})^3 + \tfrac{19}{2}(-\tfrac{1}{14})^2} + \tfrac{1}{7} - 2 = -\tfrac{1}{7} - 1.$$

Simplifying, $2\left(+\dfrac{5}{14}\right) - \dfrac{13}{7} = -\dfrac{8}{7}$, or $\dfrac{-8}{7} = \dfrac{-8}{7}.$

It is easily possible to write a statement involving radical expressions which has the *form* of an equation, but is not one. Thus $\sqrt{x+1} + \sqrt{x+3} + 1 = 0$ looks like an equation, but no value of x can satisfy it. A little closer inspection shows that the statement asserts that the sum of three positive numbers is zero, a condition clearly impossible. Statements like the one given are often called "impossible equations," though, strictly speaking, they are not equations at all. In the attempt to solve an apparent equation one may resort to the usual methods of solution and obtain a result which will not satisfy the original statement. Not until one tries to verify the result is the falsity of the original statement discovered.

3. Solve $1 + \sqrt{x + 2} = \sqrt{x}.$ (1)

Solution: Transposing, $1 - \sqrt{x} = -\sqrt{x + 2}.$ (2)

Squaring (2), $1 - 2\sqrt{x} + x = x + 2.$ (3)

Transposing and collecting,

$$-2\sqrt{x} = 1. \quad (4)$$

Squaring (4), $4\,x = 1$, or $x = \frac{1}{4}.$ (5)

Check: Substituting $\frac{1}{4}$ for x in (1),

$$1 + \sqrt{\tfrac{1}{4} + 2} = +\sqrt{\tfrac{1}{4}}.$$

$1 + \frac{3}{2} = +\frac{1}{2}$, or $\frac{5}{2} = \frac{1}{2}$, which is false.

It is fairly certain that the student did not see that the statement (1) is false until the attempt was made to verify the result. It appears, then, that the method of solution may give a result which is not a root.

4. Solve $\sqrt{x-1}+\sqrt{3x+1}-2=0.$ (1)

Solution: Transposing, $\sqrt{3x+1}=2-\sqrt{x-1}.$ (2)

Squaring both members of (2),

$$3x+1=4-4\sqrt{x-1}+x-1. \qquad (3)$$

Transposing and collecting,

$$2x-2=-4\sqrt{x-1}. \qquad (4)$$

Dividing (4) by 2, $x-1=-2\sqrt{x-1}.$ (5)

Squaring both members of (5),

$$x^2-2x+1=4x-4. \qquad (6)$$

Transposing, $x^2-6x+5=0.$ (7)

Factoring, $(x-1)(x-5)=0.$

Therefore $x=1$ or 5.

Check: Substituting 1 for x in (1),

$$\sqrt{1-1}+\sqrt{3+1}-2=0.$$
$$0+2-2=0.$$

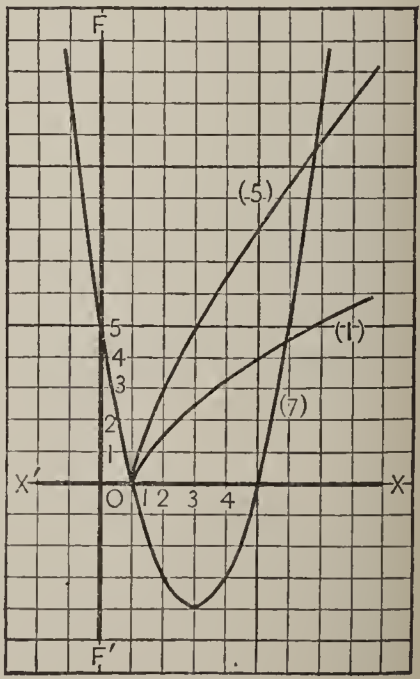
Therefore 1 is a root of (1).

Substituting 5 for x in (1),

$$\sqrt{5-1}+\sqrt{15+1}-2=0.$$
$$2+4-2=0,$$

or $4=0$; but $4\neq 0$.

Therefore 5 is not a root of (1). It was introduced by the process of squaring each member of equation (5). This process does not necessarily introduce a root. Thus 1 is a root of each of the equations (1) to (7), and while 5 is a root of (6) and (7), it is not a root of (5), as may be verified by substitution. Further, (5) was obtained by squaring (2), yet neither the root 1 nor the root 5 was introduced at that point.



Just what did happen in the course of the preceding solution is shown in the adjacent figure, where equations (1), (5), and (7) are solved graphically.

The graph shows the changes in the function due to squaring. It appears that the root 1 is common to (1), (5), and (7), while the root 5 is extraneous to (1) and (5).

As we have seen, the solution of (1) leads to the quadratic $x^2 - 6x + 5 = 0$. Since (1) and (5) have the root 1 but not 5, it is obvious that with some radical equations one may resort to squaring once without introducing an *extraneous* root.

Equation (1) is typical of many radical equations which, when solved by rationalizing, give the roots not only of the original equation, but also of such equations as may be derived from it by giving each radical therein the sign ±.

It will be seen from the next example, also, that the process of rationalization does not necessarily introduce extraneous roots.

5. Solve $\sqrt{x+2} + \sqrt{3-x} = 3$. (1)

Solution: Transposing, $\sqrt{3-x} = 3 - \sqrt{x+2}$. (2)

Squaring (2), $3 - x = 9 - 6\sqrt{x+2} + x + 2$. (3)

Transposing and collecting, $-2x - 8 = -6\sqrt{x+2}$. (4)

$(4) \div -2$, $x + 4 = 3\sqrt{x+2}$. (5)

Squaring (5), $x^2 + 8x + 16 = 9x + 18$. (6)

Transposing and collecting, $x^2 - x - 2 = 0$. (7)

Factoring, $(x-2)(x+1) = 0$.

Therefore $x = 2 \text{ or } -1$.

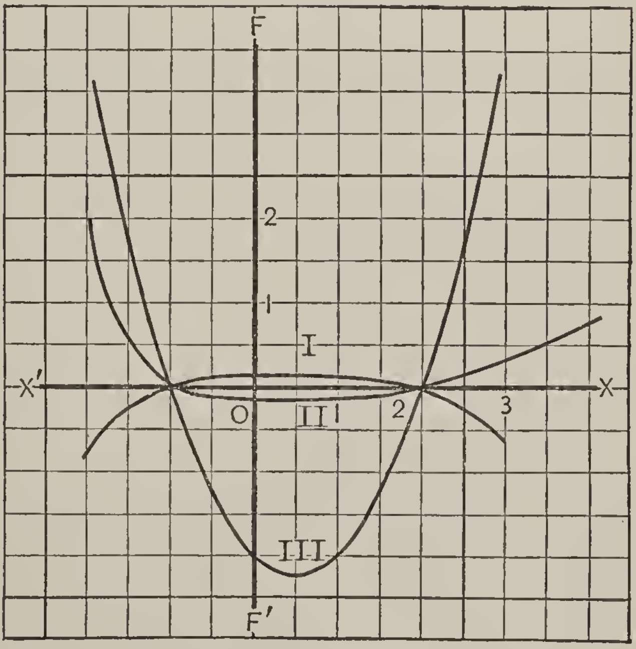
Check: Substituting 2 for x in (1),

$$\sqrt{2+2} + \sqrt{3-2} = 3, \text{ or } 2 + 1 = 3.$$

Substituting -1 for x in (1),

$$\sqrt{-1+2} + \sqrt{3+1} = 3, \text{ or } 1 + 2 = 3.$$

Therefore equation (1) has two roots, 2 and -1.



The graphical solution of (1), (5), and (7) gives curves I, II, and III respectively of the adjacent figure. These graphs show the change in the function with each resort to squaring. Curves I, II, and III intersect the x-axis at the same points, showing that the roots 2 and -1 are common to (1), (5), and (7).

It should be clear from the preceding examples that we cannot determine the number of roots of a given radical equation without solving it; nor can we predict whether the given statement involving radicals is an equation. *Results obtained are roots if they satisfy the original statement, and not otherwise.*

The method of solving a radical equation may be stated in the

Rule. *Transpose the terms so that one radical expression (the least simple one) is the only term in one member of the equation.*

Next raise both members of the resulting equation to the same power as the index of this radical.

If radical expressions still remain, repeat the two preceding operations until an equation is obtained which is free from radicals. Then solve this equation.

Check. Substitute the values found in the *original* equation and reduce the resulting radicals to their simplest form. Whenever the radicals are rational simplify by extracting the roots indicated. Never simplify by raising both members of the equation to any power, for extraneous roots introduced by that process would not then be detected.

Finally, reject all extraneous roots.

EXERCISES

Solve, check results, and reject all extraneous roots:

1. $\sqrt{x+3}=8.$

2. $\sqrt{2x-6}+4=7.$

3. $3\sqrt{2x-8}-7=17.$

4. $\sqrt[3]{3x-4}=2.$

5. $(7x+15)^{\frac{1}{3}}+18=17.$

6. $2\sqrt[3]{3n-25}+3=7.$

7. $3x\sqrt{3x}=18\sqrt{27x}.$

8. $(9x)^{\frac{1}{2}}=x^{\frac{1}{2}}+4.$

9. $\sqrt{5n^2+19}+n=7.$

10. $(3x-4)^{\frac{1}{3}}+(4x+3)^{\frac{1}{3}}=0.$

11. $4\sqrt{v^2-v-4}+3=15.$

12. $\sqrt{x+4}=\sqrt[4]{x^2-5x+6}.$

13. $\sqrt[3]{x+1}=\sqrt{x+1}.$

14. $(s-2)^{\frac{1}{2}}=s^{\frac{1}{2}}+2^{\frac{1}{2}}.$

15. $3\sqrt{r+1}-2\sqrt{r+3}=\sqrt{2r+4}-\sqrt{r+3}+2\sqrt{r+1}.$

16. $\sqrt{2x-3}+3\sqrt{x-5}=\sqrt{3x-8}+2\sqrt{x-5}.$

17. $\sqrt{4x-12}+\sqrt{5x-2}+\sqrt{9x-14}=0.$

18. $\sqrt{4x-12}-\sqrt{5x-2}-\sqrt{9x-14}=0.$

19. $\sqrt{4x-12}+\sqrt{5x-2}-\sqrt{9x-14}=0.$

20. $\sqrt{4x-12}-\sqrt{5x-2}+\sqrt{9x-14}=0.$

21. $\dfrac{7\sqrt{n}+10}{\sqrt{4n-2}}=3.$

22. $\dfrac{2\sqrt{a}}{\sqrt{2x-a}}=\dfrac{\sqrt{2x+4a}}{3\sqrt{a}}.$

23. $\dfrac{n^{\frac{1}{2}}-3}{n^{\frac{1}{2}}}-\dfrac{5-n^{\frac{1}{2}}}{4}=0.$

24. $\dfrac{\sqrt{x+16}}{\sqrt{4-x}}+\dfrac{\sqrt{4-x}}{\sqrt{x+16}}=\dfrac{5}{2}.$

25. $5r-13r^{\frac{1}{2}}+6=0.$

26. $7x-3x^{\frac{1}{2}}-2x^{\frac{3}{2}}=0.$

27. $\dfrac{(r+5)^{\frac{1}{2}}}{(r+3)^{\frac{1}{2}}}-\dfrac{(r+3)^{\frac{1}{2}}}{(r+5)^{\frac{1}{2}}}=\dfrac{2}{\sqrt{3}}.$

28. $\sqrt{7+4x+3\sqrt{2x^2+5x+7}}-3=0.$

29. $\sqrt{17+2\sqrt{3+s+\sqrt{s+7}}}-5=0.$

30. $4x^2=10x+10-2\sqrt{4x^2-10x-2}.$

31. $3m^2=6\sqrt{3m^2-m-6}+m+22.$

32. $\sqrt{x+15}+\sqrt{x-24}-\sqrt{x-13}=\sqrt{x}.$

33. Solve for l and g, $t=\pi\sqrt{\dfrac{l}{g}}.$ **34.** Solve for t, $s=\dfrac{gt^2}{2}.$

35. If $a=\dfrac{R}{2}\sqrt{2}$ and $K=2R^2$, express K in terms of a.

36. If $K=\dfrac{3R^2}{2}\sqrt{3}$ and $a=\dfrac{R}{2}\sqrt{3}$, express K in terms of a.

37. If $K=2r^2\sqrt{2}$ and $a=\dfrac{r}{2}\sqrt{2+\sqrt{2}}$, express K in terms of a.

38. If $K=3r^2$ and $a=\dfrac{r}{2}\sqrt{2+\sqrt{3}}$, express K in terms of a.

39. The perimeter and the area of a certain square exceed the perimeter and area of a second square by 72 feet and 900 square feet respectively. Find the side of each square.

40. If a bullet is fired vertically upward, the least velocity V which it may have so that it will never return to the earth is given by the equation $V = \sqrt{2gR}$. ($g = 32$ feet per second, $R = 4000$ miles.) Find the velocity in miles per second to the nearest whole number.

41. The greatest distance x (in feet) that a ball can be thrown with velocity v (in feet per second) across a level field is given by one root of the equation $.976\,v^2x - gx^2 = 0$. ($g = 32$.) Under the conditions just stated a ball is thrown with a velocity of 100 feet per second. How far from the thrower does it strike the ground?

42. The greatest distance a baseball has been thrown is 426 feet $6\frac{1}{4}$ inches (Sheldon Lejeune, October 10, 1910). With what velocity did it leave the thrower's hand? (This velocity is called the initial velocity.)

43. Determine the initial velocity from the data: (*a*) A Lacrosse ball has been thrown 497 feet $7\frac{1}{2}$ inches (B. Quinn, 1902). (*b*) The record distance for the 16-pound shot is 51 feet (Ralph Rose, 1909). (*c*) The 16-pound hammer has been thrown 184 feet 4 inches (John Flanagan, 1910). (*d*) A football has been kicked a distance of 200 feet (W. P. Chadwick, 1887).

CHAPTER XXXIII

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

153. Graph of a quadratic equation in two variables. Before solving graphically a quadratic system, the method of graphing *one* quadratic equation in *two* variables must be clearly understood.

EXAMPLES

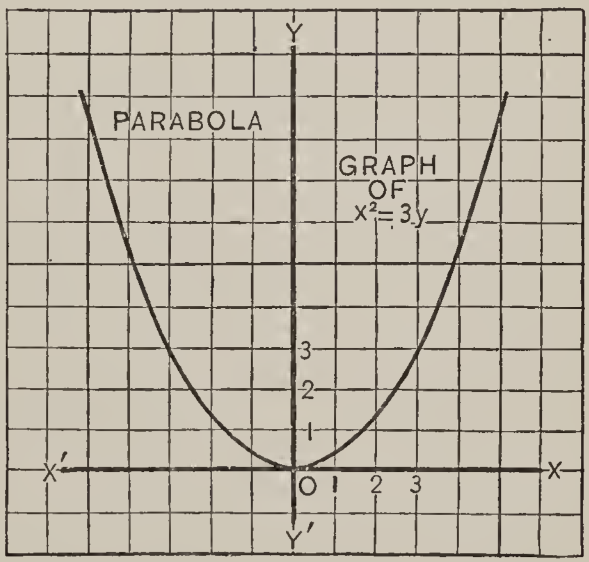
1. Construct the graph of $x^2 = 3y$.

Solution: Solving the equation for x in terms of y, $x = \pm\sqrt{3y}$.

We now assign values to y and then compute the approximate corresponding values of x. Tabulating the results gives:

$y =$	9	4	3	2	1	0	-1	Any negative value
$x =$	± 5.19	± 3.46	± 3	± 2.44	± 1.73	0	$\pm\sqrt{-3}$	Imaginary

Using an x-axis and a y-axis as in graphing linear equations, plotting the points corresponding to the real numbers in the table, and drawing the curve determined by these points, we obtain the graph of the adjacent figure. Since y is a function of x, the y-axis corresponds to the *function* axis. The curve is a **parabola**. A similar curve was always obtained in Chapter VII for the graph of a quadratic function of one variable.



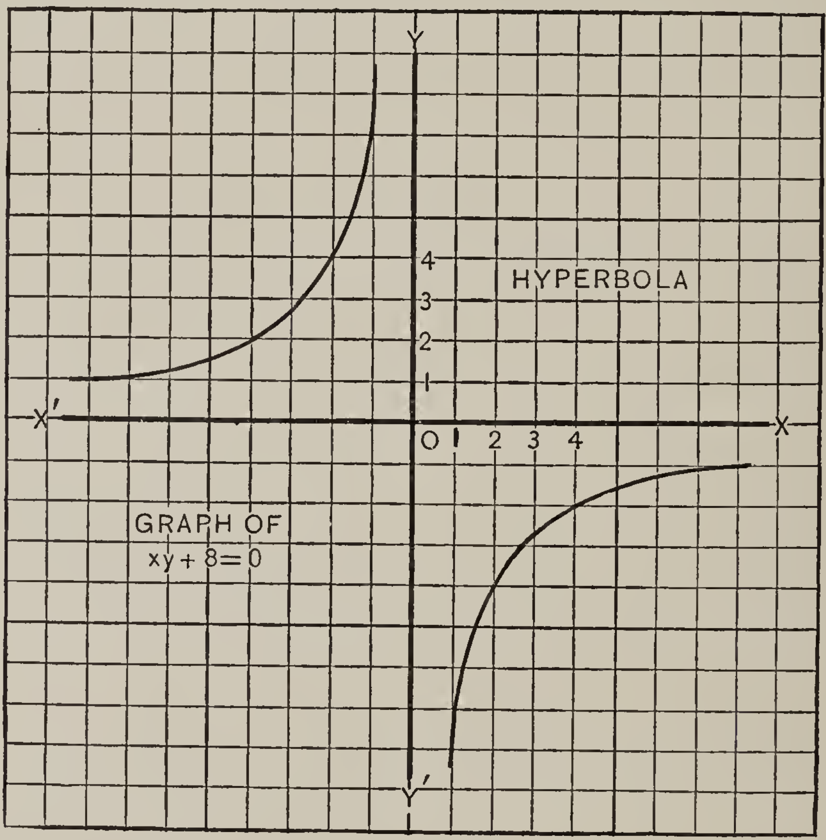
The graph of any equation of the form $y^2 = ax$ is a *parabola.*

2. Graph the equation $xy + 8 = 0$.

Solution: Solving for y in terms of x, $y = -\frac{8}{x}$.

Assigning values to x as indicated in the following table, we then compute the corresponding values of y.

$x =$	-6	-5	-4	-3	-2	-1	$-\frac{3}{4}$	$\frac{3}{4}$	1	2	3	4	5	6	8
$y =$	$\frac{4}{3}$	$\frac{8}{5}$	2	$\frac{8}{3}$	4	8	$\frac{32}{3}$	$-\frac{32}{3}$	-8	-4	$-\frac{8}{3}$	-2	$-\frac{8}{5}$	$-\frac{4}{3}$	-1



Proceeding as before with the numbers in the table, we obtain the two-branched curve of the above figure, which does not touch either axis. The curve is called an **hyperbola.**

The graph of any equation of the form $xy = K$ is an *hyperbola.* The curve for $xy = K$ (K = any constant) is always in the same general position. That is, if K is positive, one branch of the curve lies in the first quadrant and the other branch in the third. If K is negative, one branch lies in the second quadrant and the other in the fourth.

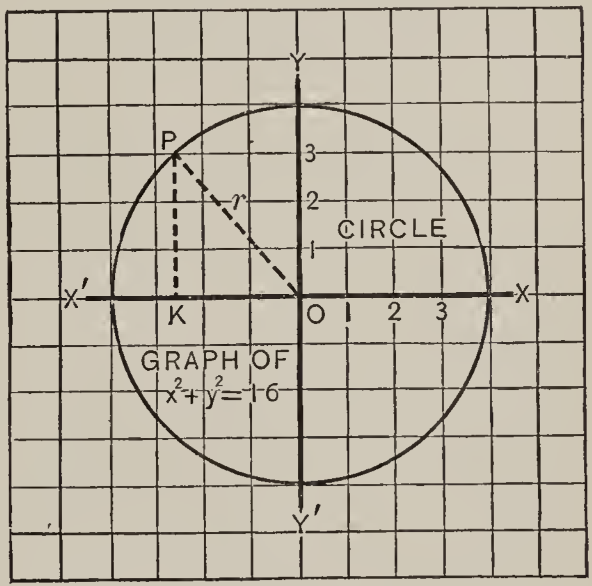
3. Graph the equation $x^2 + y^2 = 16$.

Solution: Solving for y in terms of x, $y = \pm\sqrt{16 - x^2}$.

Assigning values to x as indicated in the following table, we compute the corresponding approximate values of y.

$x =$	5	-4	-3	-2	-1	0	1	2	3	4	5
$y =$	$\pm 3\sqrt{-1}$	0	± 2.64	± 3.46	± 3.87	± 4	± 3.87	± 3.46	± 2.64	0	$\pm 3\sqrt{-1}$

For values of x numerically greater than 4, y is *imaginary*. The points corresponding to the pairs of real numbers in the table lie on the **circle** in the adjacent figure. The center of the circle is at the origin and the radius is 4.



Further, the graph of any equation of the form

$$x^2 + y^2 = r^2$$

is a *circle* whose radius is r. This can be proved from the right triangle PKO. If P represents *any* point on the circle, OK equals the x-distance of P, KP equals the y-distance, and OP equals the radius. Now $\overline{OK}^2 + \overline{KP}^2 = \overline{OP}^2$; that is, $x^2 + y^2 = r^2$. It follows, then, that the graphs of $x^2 + y^2 = 9$ and $x^2 + y^2 = 8$ are circles whose centers are at the origin and whose radii are 3 and $\sqrt{8}$ respectively. Hereafter, when it is required to graph an equation of the form $x^2 + y^2 = r^2$, the student may use compasses, and, with the origin as the center and the proper radius (the square root of the constant term), describe the circle at once.

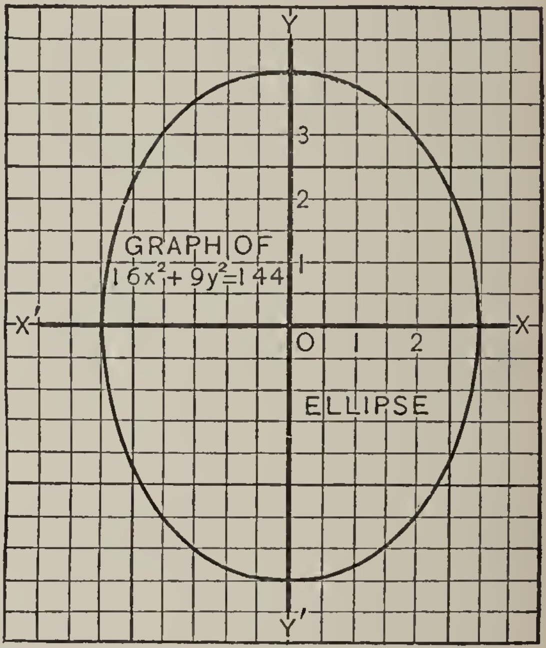
In all of the graphical work which follows it is expected that the student will save time by obtaining from the curve on page 85, or from the table on page 501, the square roots or cube roots which he may need.

4. Graph the equation $16x^2 + 9y^2 = 144$.

Solution: Solving for y in terms of x, $y = \pm \frac{4}{3}\sqrt{9 - x^2}$.

Assigning values to x as indicated in the following table, we compute the corresponding approximate values of y.

$x=$	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
$y=$	$\pm\frac{4}{3}\sqrt{-7}$	0	± 2.98	± 3.77	± 4	± 3.77	± 2.98	0	$\pm\frac{4}{3}\sqrt{-7}$



For values of x numerically greater than 3, y is *imaginary*. The points corresponding to the real numbers in the table lie on the graph of the adjacent figure. The curve is called an **ellipse**.

The graph of any equation of the form of $ax^2 + by^2 = c$, in which a and b are unequal and of the same sign as c, is an *ellipse*.

Note. These three curves, the ellipse, the hyperbola, and the parabola, were first studied by the Greeks, who proved that they are the sections which one obtains by cutting a cone by a plane. Not for hundreds of years afterwards did any one imagine that these curves actually appear in nature, for the Greeks regarded them merely as geometrical figures, and not at all as curves that have anything to do with our everyday life. One of the most important discoveries of astronomy was made by Kepler (1571–1630), who showed that the earth revolves around the sun in an ellipse, and stated the laws which govern the motion. Those comets that return to our field of vision periodically also have elliptic orbits, while those that appear once, never to be seen again, describe parabolic or hyperbolic paths.

The path of a ball thrown through the air in any direction, except vertically upward or downward, is a parabola. The approximate parabola which a projectile actually describes depends on the elevation of the gun (the angle with the horizontal), the quality of the powder, the amount of the charge, the direction of the wind, and various other conditions. This makes gunnery a complex problem.

EXERCISES

Construct the graphs of the following equations and state the name of each curve obtained:

1. $x^2 = 2y$.
2. $y^2 + 2x = 0$.
3. $x^2 + y^2 = 36$.
4. $x^2 + y^2 = 12$.
5. $x^2 - y^2 = 25$.
6. $xy = 8$.
7. $xy = -12$.
8. $9x^2 + 16y^2 = 144$.
9. $16x^2 - 9y^2 = 144$.
10. $25x^2 + 9y^2 = 225$.

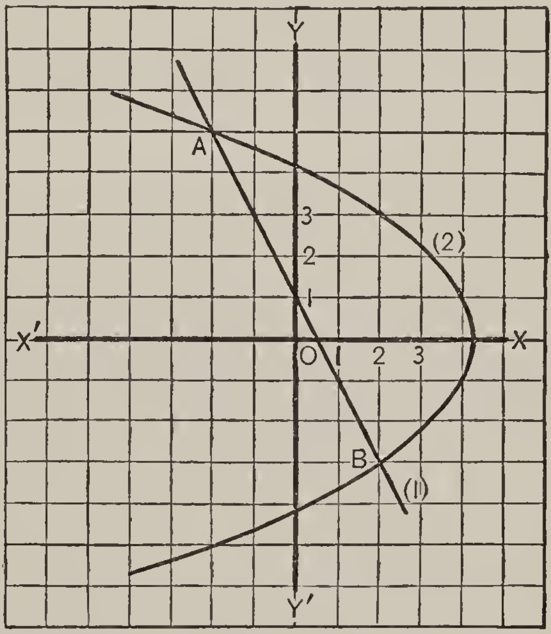
154. Graphical solution of a quadratic system in two variables. That we may solve a system of two quadratic equations by a method similar to that employed in § 81 for linear equations appears from the following

EXAMPLES

1. Solve graphically $\begin{cases} 2x + y = 1, & (1) \\ y^2 + 4x = 17. & (2) \end{cases}$

Solution: Constructing the graphs of (1) and (2), we obtain the straight line and the parabola shown in the adjacent figure. There are two sets of roots corresponding to the two points of intersection, which are:

$$A\begin{cases} x = -2, \\ y = 5, \end{cases} \quad B\begin{cases} x = 2, \\ y = -3. \end{cases}$$



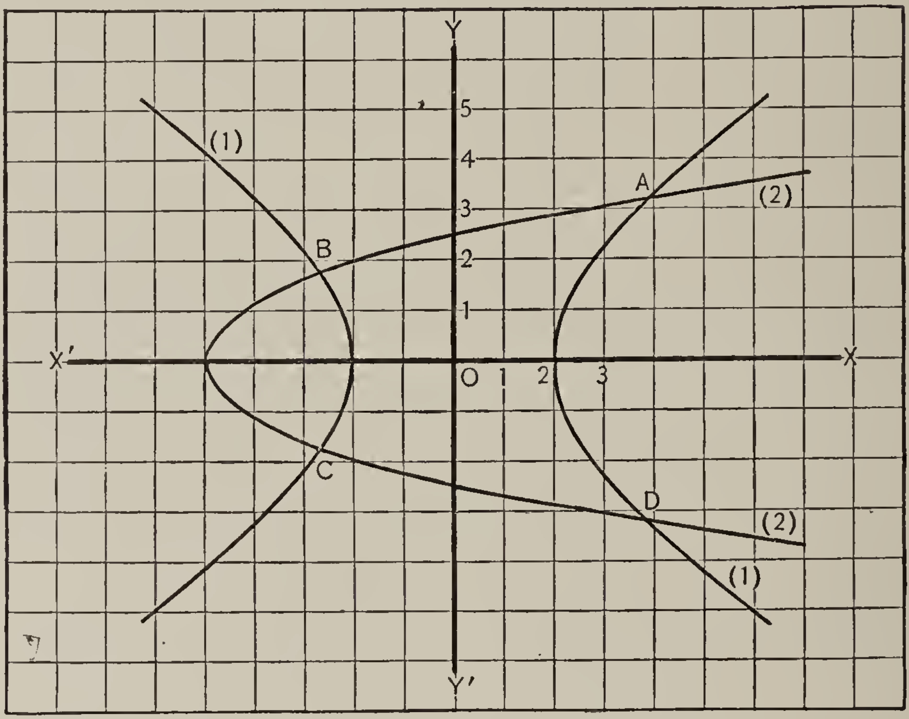
Note. If the straight line in the adjacent figure were moved to the right in such a way that it always remained parallel to its present position, the points A and B would approach each other and finally coincide. The line would then be tangent to the parabola at the point $x = 4$, $y = 1$.

Were the straight line moved still farther, it would neither touch nor intersect the parabola and there would be no graphical solution.

An illustration of these two conditions is given by the graphical solution of Exercises 8 and 9, page 390.

2. Solve graphically $\begin{cases} x^2 - y^2 = 4, & (1) \\ y^2 - x - 6 = 0. & (2) \end{cases}$

Solution: Constructing the graphs of (1) and (2), we obtain the hyperbola and the parabola of the following figure. There are four sets of roots corresponding to the four points of intersection, which are approximately



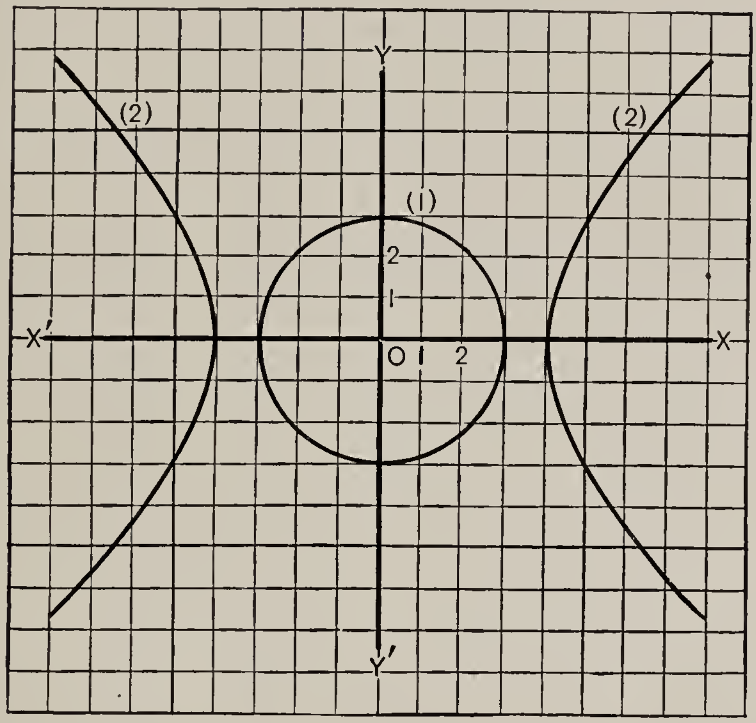
$$A\begin{cases} x = 3.7, \\ y = 3.1. \end{cases} \quad B\begin{cases} x = -2.7, \\ y = 1.8. \end{cases} \quad C\begin{cases} x = -2.7, \\ y = -1.8. \end{cases} \quad D\begin{cases} x = 3.7, \\ y = -3.1. \end{cases}$$

3. Solve graphically $\begin{cases} x^2 + y^2 = 9, & (1) \\ x^2 - y^2 = 16. & (2) \end{cases}$

Solution: The graphs (1) and (2) are the circle and hyperbola of the figure on page 389. These curves have no real points of intersection. There are, however, four sets of imaginary roots. Subtracting equation (2) from (1) gives $2y^2 = -7$, whence $y = \pm\sqrt{-\frac{7}{2}}$, an *imaginary* expression. Adding (1) and (2) gives $2x^2 = 25$, whence $x = \pm\frac{5}{2}\sqrt{2}$, a real expression. Using the double sign before each radical gives the four sets of imaginary roots:

$$\begin{cases} x = \frac{5}{2}\sqrt{2}, \\ y = \sqrt{-\frac{7}{2}}, \end{cases} \quad \begin{matrix} +\frac{5}{2}\sqrt{2}, \\ -\sqrt{-\frac{7}{2}}, \end{matrix} \quad \begin{matrix} -\frac{5}{2}\sqrt{2}, \\ -\sqrt{-\frac{7}{2}}, \end{matrix} \quad \begin{matrix} -\frac{5}{2}\sqrt{2}. \\ +\sqrt{-\frac{7}{2}}. \end{matrix}$$

It can be shown that these sets of imaginary roots correspond to the intersections of what may be termed the imaginary branches of the curves. These branches may be represented as lines not in the same plane as the real branches but in a plane passing through the x-axis perpendicular to the plane determined by the x-axis and the y-axis.



Though the subject is not difficult, even a simple presentation of this method of constructing imaginary graphs is wholly beyond the scope of this book. The essential point to be grasped now is that real roots correspond to real intersections, and imaginary roots correspond to no intersections of real graphs.

Note. In equation (1), page 388, a greater number in place of 9 would give a larger circle than the one in the figure, and it would be easy to find a number to replace 9 such that the resulting circle would just touch the hyperbola. Were a still greater number used, the circle obtained would intersect the other curve. These varying conditions would result, respectively, in (a) no set of real roots, (b) two sets of real roots, (c) four sets of real roots.

Examples 1, 2, and 3 partially illustrate the truth of the following statement:

If in a system of two equations in two variables one equation is of the mth degree and one of the nth, there are *usually* **mn** sets of roots (real or imaginary) *and never more than* **mn** *such sets.*

EXERCISES

If possible, solve graphically each of the following systems:

1. $\begin{array}{l} x^2 = 4y, \\ x + 3y = 5. \end{array}$

2. $\begin{array}{l} x^2 + y^2 = 25, \\ x - 2y = 10. \end{array}$

3. $\begin{array}{l} x^2 + y^2 = 16, \\ x^2 + y^2 = 9. \end{array}$

4. $\begin{array}{l} x^2 + y^2 = 4, \\ x + y = 8. \end{array}$

5. $\begin{array}{l} x^2 + y^2 = 25, \\ x^2 - y^2 = 16. \end{array}$

6. $\begin{array}{l} x^2 + y^2 = 16, \\ x^2 - y^2 = 25. \end{array}$

7. $\begin{array}{l} y^2 = 4x, \\ x^2 + 9y^2 = 9. \end{array}$

8. $\begin{array}{l} y^2 + 4x = 17, \\ 2x + y = 9. \end{array}$

9. $\begin{array}{l} y^2 + 4x = 17, \\ 2x + y = 12. \end{array}$

10. $\begin{array}{l} x^2 + y^2 = 16, \\ x^2 + y^2 - 2x = 8. \end{array}$

11. $\begin{array}{l} y - x\sqrt{8} = 0, \\ y^2 = x^3 - 9x. \end{array}$

CHAPTER XXXIV

SYSTEMS SOLVABLE BY QUADRATICS

155. Introduction. The general equation of the second degree in two variables is $ax^2 + by^2 + cxy + dx + ey + f = 0$. To solve a pair of such equations requires the solution of an equation of the fourth degree. Even the solution of $x^2 + y = 5$ and $y^2 + x = 3$ requires the solution of a biquadratic equation. In fact, only a limited number of systems of the second degree in two variables are solvable by quadratics. The student should note that he can solve graphically for real roots any system of quadratic equations, provided the terms have numerical coefficients. The algebraic solution of such systems will be possible for him only after further study of algebra.

156. Linear and quadratic. Every system of equations in two variables in which one equation is **linear** and the other **quadratic** *can be solved by the method of substitution.*

EXAMPLE

Solve the system $\begin{cases} x^2 + y^2 = 5, & (1) \\ x - y = 1. & (2) \end{cases}$

Solution: Solving (2) for x in terms of y, $x = 1 + y$. (3)

Substituting $1 + y$ for x in (1), $(1 + y)^2 + y^2 = 5$. (4)

From (4), $y^2 + y - 2 = 0$. (5)

Solving (5), $y = 1$ or -2.

Substituting 1 for y in (3), $x = 1 + 1 = 2$.

Substituting -2 for y in (3), $x = 1 - 2 = -1$.

The two sets of roots are $x = 2$, $y = 1$ and $x = -1$, $y = -2$.

Check: Substituting 2 for x and 1 for y in $\begin{cases} (1), & 4 + 1 = 5, \\ (2), & 2 - 1 = 1. \end{cases}$

Substituting -1 for x and -2 for y in $\begin{cases} (1), & 1 + 4 = 5, \\ (2), & -1 + 2 = 1. \end{cases}$

EXERCISES

Solve the following systems, pair results, and check each set of roots:

1. $x + y = 6,$
$x^2 + y^2 = 20.$

2. $4m + n = 28,$
$2m^2 + 3mn = 98.$

3. $m^2 + 2n^2 = 44,$
$m - 2n\sqrt{5} = 0.$

4. $4s + t = 6,$
$st = -10.$

5. $xy + 36 = 0,$
$4x - y = 30.$

6. $x\sqrt{3} + 5y = -72,$
$xy = -15\sqrt{3}.$

7. $3R_1 + 4R_2 = 5,$
$2R_1R_2 - 6R_1 = -3.$

8. $2xy + y^2 - 20 = 0,$
$xy + 40 = 0.$

9. $h^2 + k^2 + 2k = 40,$
$h + k + 2 = 0.$

10. $m^2 + 3mn + n^2 = 88,$
$2m = n.$

11. $x^2 + y^2 + 4x + 6y = 40,$
$x - 10 = y.$

12. $y + x\sqrt{15} = 0,$
$y^2 + x^3 = 16x.$

If the equations of a system are not one *linear* and the other *quadratic*, an attempt to solve it by *substitution* usually gives an equation of the third or fourth degree at least. In most cases such an equation could not be solved by factoring, and at the present time its solution by any other method is beyond the student. The various devices explained in the following pages are for the purpose of avoiding the necessity of solving an equation of a higher degree than the second.

157. Homogeneous equations. An equation is **homogeneous** if, on being written so that one member is zero, the terms in the other member are of the same degree with respect to the variables.

Thus $x^2 + y^2 = xy$ and $x^2 - 3xy + y^2 = 0$ are homogeneous equations of the second degree; $2x^3 + y^3 = x^2y - 3xy^2$ is a homogeneous equation of the third degree.

158. Both equations quadratic. If the system is of either type described in the following examples, it can be solved by quadratics.

The first example illustrates the type when one equation, but not necessarily both of them, is homogeneous.

EXAMPLES

1. Solve the system $\begin{cases} 3x^2 + 4y^2 = 8xy, & (1) \\ y^2 + x^2 - 5x = 3. & (2) \end{cases}$

Solution: First we solve the homogeneous equation (1) for x in terms of y.

Transposing in (1), $3x^2 - 8xy + 4y^2 = 0.$ (3)

Solving (3) by formula, $x = \dfrac{8y \pm \sqrt{64y^2 - 48y^2}}{6}.$ (4)

Whence $x = 2y$ or $\dfrac{2y}{3}.$ (5)

Substituting $2y$ for x in (2),

$$y^2 + 4y^2 - 10y = 3. \quad (6)$$

Solving (6), $y = 1 \pm \frac{2}{5}\sqrt{10}.$ (7)

By (5), $x = 2y$; then from (7), $x = 2 \pm \frac{4}{5}\sqrt{10}.$ (8)

Substituting $\dfrac{2y}{3}$ for x in (2),

$$y^2 + \frac{4y^2}{9} - \frac{10y}{3} = 3. \quad (9)$$

Solving (9), $y = 3$ or $-\frac{9}{13}.$ (10)

By (5), $x = \dfrac{2y}{3}$; then from (10), $x = 2$ or $-\frac{6}{13}.$ (11)

Pairing results,

$$\left.\begin{matrix} x = 2 \\ y = 3 \end{matrix}\right\} A, \quad \left.\begin{matrix} x = -\frac{6}{13} \\ y = -\frac{9}{13} \end{matrix}\right\} B, \quad \left.\begin{matrix} x = 2 + \frac{4}{5}\sqrt{10} \\ y = 1 + \frac{2}{5}\sqrt{10} \end{matrix}\right\} C, \quad \left.\begin{matrix} x = 2 - \frac{4}{5}\sqrt{10} \\ y = 1 - \frac{2}{5}\sqrt{10} \end{matrix}\right\} D.$$

Check:

$$A\begin{cases} 3(2)^2 + 4(3)^2 = 8 \cdot 2 \cdot 3, \text{ or } 48 = 48. \\ 3^2 + 2^2 - 5 \cdot 2 = 3, \text{ or } 3 = 3. \end{cases}$$

$$B\begin{cases} 3(-\frac{6}{13})^2 + 4(-\frac{9}{13})^2 = 8 \cdot (-\frac{6}{13})(-\frac{9}{13}), \text{ or } \frac{432}{169} = \frac{432}{169}. \\ (-\frac{9}{13})^2 + (-\frac{6}{13})^2 - 5 \cdot (-\frac{6}{13}) = 3, \text{ or } 3 = 3. \end{cases}$$

$$\begin{matrix} C \\ D \end{matrix}\begin{cases} 3(2 \pm \frac{4}{5}\sqrt{10})^2 + 4(1 \pm \frac{2}{5}\sqrt{10})^2 = 8(2 \pm \frac{4}{5}\sqrt{10})(1 \pm \frac{2}{5}\sqrt{10}). \\ (1 \pm \frac{2}{5}\sqrt{10})^2 + (2 \pm \frac{4}{5}\sqrt{10})^2 - 5(2 \pm \frac{4}{5}\sqrt{10}) = 3. \end{cases}$$

Taking both values in C and D with the sign $+$ or both with the sign $-$,

$$\begin{cases} 12 \pm \frac{48}{5}\sqrt{10} + \frac{96}{5} + 4 \pm \frac{16}{5}\sqrt{10} + \frac{32}{5} = 16 \pm \frac{64}{5}\sqrt{10} + \frac{128}{5}. \\ 1 \pm \frac{4}{5}\sqrt{10} + \frac{8}{5} + 4 \pm \frac{16}{5}\sqrt{10} + \frac{32}{5} - 10 \mp 4\sqrt{10} = 3. \end{cases}$$

If each equation of a system in two variables is quadratic and both are homogeneous with the exception of a constant term (not zero), the system is solved much like the preceding one.

2. Solve $\begin{cases} xy + 3y^2 = 6, & (1) \\ x^2 + y^2 = 10. & (2) \end{cases}$

Hint. First we combine the two equations to obtain a homogeneous equation in which the constant term is zero.

$(1) \cdot 5,$ $\quad 5xy + 15y^2 = 30.$ $\quad (3)$
$(2) \cdot 3,$ $\quad 3x^2 + 3y^2 = 30.$ $\quad (4)$
$(3) - (4),$ $\quad -3x^2 + 5xy + 12y^2 = 0.$ $\quad (5)$

Solving (5) for x in terms of y, $x = 3y$ or $-\frac{4y}{3}$. $\quad (6)$

We can now substitute from (6) in (2) and proceed precisely as in the last example. The student should complete the work and obtain

$$\begin{array}{llll} x = 3, & -3, & +\frac{4}{5}\sqrt{10}, & -\frac{4}{5}\sqrt{10}. \\ y = 1, & -1, & -\frac{3}{5}\sqrt{10}, & +\frac{3}{5}\sqrt{10}. \end{array}$$

EXERCISES

Solve, pair results, and check each set of real roots:

1. $x^2 + xy = 3,$
$y^2 - xy = 10.$

2. $x^2 + y^2 = 10,$
$3y^2 + xy = 6.$

3. $u^2 + 2uv = 0,$
$2v^2 + 3uv = -16.$

4. $s^2 - 3st = 4,$
$3t^2 + 3s^2 = 12.$

5. $x(x + 2y) = 16,$
$y(y - x) = 3.$

6. $x^2 + xy + y^2 = 4,$
$x^2 - 2xy = 12.$

7. $x^2 + xy = 2y^2,$
$2x^2 + x = 2 + y^2.$

8. $x^2 + 2xy - y^2 = 32,$
$2x^2 - 3xy + y^2 = 0.$

9. $2x^2 - xy + 2y^2 = 12,$
$2x^2 + xy + 2y^2 = 8.$

10. $x^2 - xy - 5y^2 = 15,$
$x^2 - 6y^2 = 1.$

Up to this point the systems considered have been solved by a method partially described by the word "substitution." The essential step in this method is to solve one of the original equations (or one derived from the original system) for one variable in terms of the other, and substitute the value found in the other equation (or in either of the original equations). This method is applicable more frequently than those which are given later. Consequently it is much more important for the student to master the method of substitution than it is for him to master any other method.

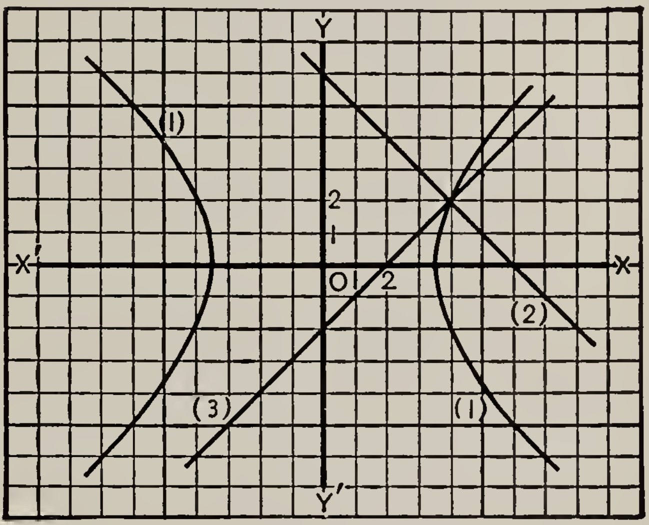
159. Equivalent systems. Equivalent systems of equations are systems which have the same set or sets of roots.

The graphs of equivalent systems have common points of intersection.

If we solve graphically the two systems

$$A\begin{cases} x^2 - y^2 = 12, & (1) \\ x + y = 6. & (2) \end{cases} \quad \text{and} \quad B\begin{cases} x - y = 2, & (3) \\ x + y = 6. & (4) \end{cases}$$

we obtain the graphs of the adjacent figure. The hyperbola (1) and the straight line (2) intersect at only one point (4, 2). The straight lines (2) and (3) intersect at this very point. Hence the systems A and B are equivalent.



160. Special devices. Systems of equations are often met which can be solved by substitution, but which are more conveniently solved as in the following illustrations. It should be observed that in every case the aim of the device is to replace the given system by an equivalent system of linear equations, or by a system in which one equation is quadratic and the other linear.

EXAMPLES

1. Solve the system $\begin{cases} x + y = 7, & (1) \\ xy = 6. & (2) \end{cases}$

Solution: Squaring (1), $x^2 + 2xy + y^2 = 49.$ (3)

(2)·4, $4xy = 24.$ (4)

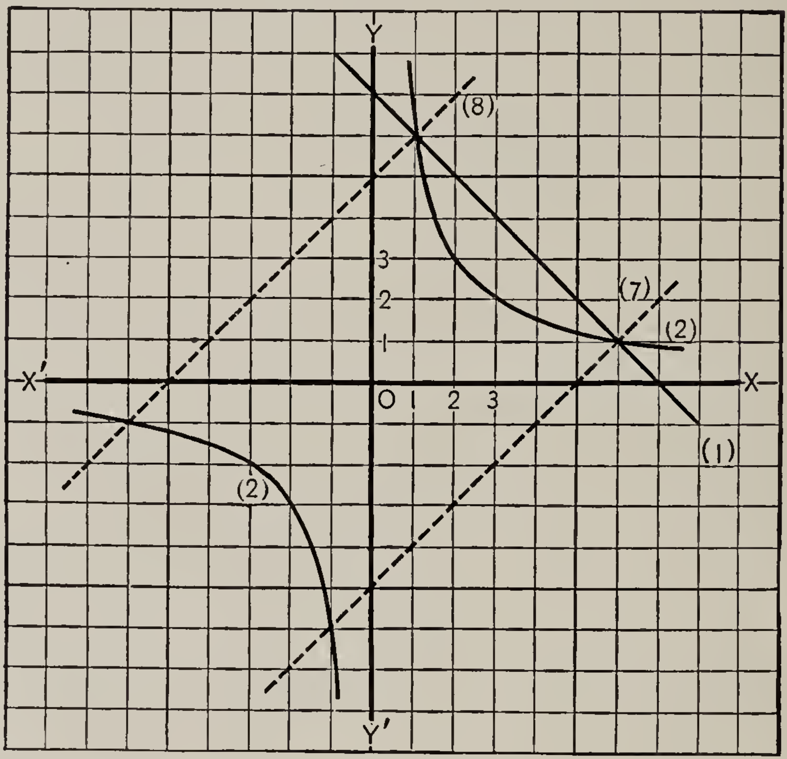
(3) − (4), $x^2 - 2xy + y^2 = 25.$ (5)

From (5), $x - y = \pm 5.$ (6)

From (6) and (1), $A\begin{cases} x + y = 7, & (1) \\ x - y = 5. & (7) \end{cases}$ $B\begin{cases} x + y = 7, & (1) \\ x - y = -5. & (8) \end{cases}$

For A, $x = 6$, $y = 1$; and for B, $x = 1$, $y = 6$.

The derived systems A and B are equivalent to the original system (1), (2). The graphs of the adjacent figure show that the straight line (1) and the hyperbola (2) have the same points of intersection as the three straight lines (1), (7), and (8) of systems A and B.



A method similar to that of the preceding solution can be applied to the following system:

2. Solve $\begin{cases} x^2 + y^2 = 37, & (1) \\ xy = 6. & (2) \end{cases}$

Solution: (2) · 2, $2xy = 12.$ (3)

(1) + (3), $x^2 + 2xy + y^2 = 49.$ (4)

From (4), $x + y = \pm 7.$ (5)

(1) − (3), $x^2 - 2xy + y^2 = 25.$ (6)

From (6), $x - y = \pm 5.$ (7)

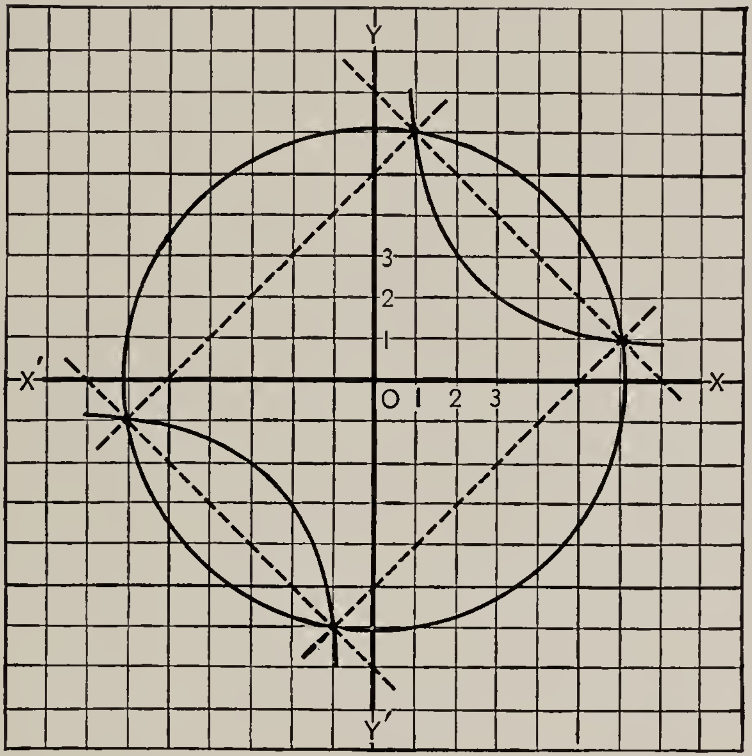
(5) and (7) combined give four systems of equations:

$A\begin{cases} x + y = 7, & (8) \\ x - y = 5. & (9) \end{cases}$ $C\begin{cases} x + y = -7, & (11) \\ x - y = 5, & (9) \end{cases}$

$B\begin{cases} x + y = 7, & (8) \\ x - y = -5. & (10) \end{cases}$ $D\begin{cases} x + y = -7, & (11) \\ x - y = -5. & (10) \end{cases}$

The solution of A, B, C, and D is left to the student.

In the figure on page 397 the graphs of (1) and (2) are the circle and the hyperbola respectively, the two curves having four points of intersection. The graphs of the four equations in the systems *A*, *B*, *C*, and *D* are the four straight lines. These four straight lines intersect in the four points in which the hyperbola and the circle intersect. This shows that the four sets of roots belonging to the system (1), (2) are identical with the four sets belonging to the four systems *A*, *B*, *C*, and *D*; that is, the *one* system, (1) (2), is *equivalent* to the *four* systems *A*, *B*, *C*, and *D*.



EXERCISES

Solve in a manner similar to that of the two preceding examples, pair results, and check each set of real roots :

1. $$\begin{aligned} x - y &= 4, \\ xy &= 5. \end{aligned}$$

2. $$\begin{aligned} x + 2y &= 8, \\ xy + 6 &= 0. \end{aligned}$$

3. $$\begin{aligned} x^2 + 4y^2 &= 101, \\ xy + 5 &= 0. \end{aligned}$$

4. $$\begin{aligned} 6x - y &= 24, \\ 36x^2 + y^2 &= 288. \end{aligned}$$

5. $$\begin{aligned} 4x^2 - 6xy + 0y^2 &= 24, \\ xy - 20 &= 0. \end{aligned}$$

6. $$\begin{aligned} 4x^2 + y^2 &= 25, \\ 4x^2 + 4xy + y^2 &= 49. \end{aligned}$$

7. $$\begin{aligned} x^2 + 4y^2 &= 15, \\ x + 2y &= 3\sqrt{3}. \end{aligned}$$

8. $$\begin{aligned} x^2 - 2xy &= 16, \\ 2y^2 - xy &= -6. \end{aligned}$$

9. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=13,$
$\dfrac{1}{xy}=6.$

10. $\dfrac{1}{x^2}-\dfrac{1}{xy}+\dfrac{1}{y^2}=7,$
$\dfrac{1}{x}-\dfrac{1}{y}=1.$

11. $3x-3y=7,$
$9x^2+18xy+9y^2=1.$

12. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=35,$
$\dfrac{1}{x}+\dfrac{1}{y}=5.$

13. $x^2+4y^2=c,$
$4xy=d.$

161. Use of division in equations. Sometimes an equation simpler than either of those given can be derived from a system by dividing the left and right members of the first equation by the corresponding members of the second. Then the equation so obtained taken with one of the first two gives a derived system more simple than the original one but not always equivalent to it. The conditions under which the two are equivalent, however, is easily stated and explained.

THEOREM. *Let U, V, K, and R be rational integral expressions in two unknowns, x and y. Then the system*

$$UK=VR, \qquad (1)$$
$$U=V \qquad (2)$$

is equivalent to the two systems

$$\begin{array}{ll} K=R, & (3) \\ U=V, & (2) \end{array} \quad \text{and} \quad \begin{array}{ll} U=0, & (4) \\ V=0. & (5) \end{array}$$

Proof. Substituting U for V in (1), transposing, and factoring, gives

$$U(K-R)=0. \qquad (6)$$

From (2), $$U-V=0. \qquad (7)$$

But the system (6), (7) is equivalent to the two systems (3), (2) and (4), (5). This is at once apparent since it can be seen from inspection that any set of roots which satisfies (3), (2) or (4), (5) will satisfy (6), (7); and conversely.

Now if U or V is an *arithmetical number* (not zero), the system (3), (2) *alone* is equivalent to the original one, since either (4) or (5) would not in that case involve any unknown.

Therefore in a system of the form of (1), (2) we may use division and thereby obtain one simpler equivalent system *if U or V is an arithmetical number.* In any other case we can at once write down the *two* systems which are equivalent to the original one. Either of these courses makes it easier to obtain all the sets of roots which satisfy the original system.

EXAMPLES

In Examples 1, 2, and 3 division gives in each case the *one* equivalent system on the right.

1. $$\begin{array}{lll} x^2-y^2=12, & x-y=2, & \text{(See graph, p. 395.)} \\ x+y=6. & x+y=6. & \text{(One set of roots.)} \end{array}$$

2. $$\begin{array}{lll} x^2-y^2=4x+6y-8, & x+y=2x+3y-4, & \\ x-y=2. & x-y=2. & \text{(One set of roots.)} \end{array}$$

3. $$\begin{array}{lll} x^3+y^3=28, & x^2-xy+y^2=7. & \text{(See graph, p. 400.)} \\ x+y=4. & x+y=4. & \text{(Two sets of roots.)} \end{array}$$

4. $$\begin{array}{l} x^3-y^3=6x+3y-18, \\ x-y=2x+y-6. \end{array}$$ Division gives the *two* systems:

$$\begin{cases} x^2+xy+y^2=3, \\ x+2y=6, \end{cases} \text{and} \begin{cases} x-y=0, \\ 2x+y-6=0. \end{cases} \text{(Three sets of roots.)}$$

The first system in Example 3 has *two* sets of roots, that in Example 4 has *three.* Hence the use of division without a correct use of the theorem on page 398 would frequently result in an incomplete solution. If time permits, the student should graph the equations of Example 4.

EXERCISES

Solve (using division where possible), pair results, and check each set of real roots:

1. $$\begin{array}{l} 4x^2-y^2=16, \\ 2x+y=8. \end{array}$$

2. $$\begin{array}{l} R^2h-75=0, \\ Rh=15. \end{array}$$

3. $$\begin{array}{l} \dfrac{1}{x^2}-\dfrac{1}{y^2}=15, \\ \dfrac{1}{x}+\dfrac{1}{y}=3. \end{array}$$

4. $9h^2k - 100 = 0,$ $3k^2h + 80 = 0.$

5. $P(1 + r)^2 = 224.72,$ $P + Pr = 212.$

6. $9x^2y^2 + 6 = 15xy,$ $3xy + 4 = 6.$

7. $x^4 = 9y^4 + 48,$ $x^2 = 3y^2 + 2.$

8. $\frac{gt^2}{2} = .16,$ $gt = 3.2.$

9. $1 - x = y,$ $1 - x^3 = y^2.$

10. $x^2 - 2xy - 24y^2 = 32,$ $x - 6y = 2.$

11. $4x^2 + 7 = 8xy + 5y^2,$ $4x^2 = 1 + y^2.$

12. $x^3 + y^3 = 4x - 6y - 8,$ $x + y = 2x - 3y - 4.$

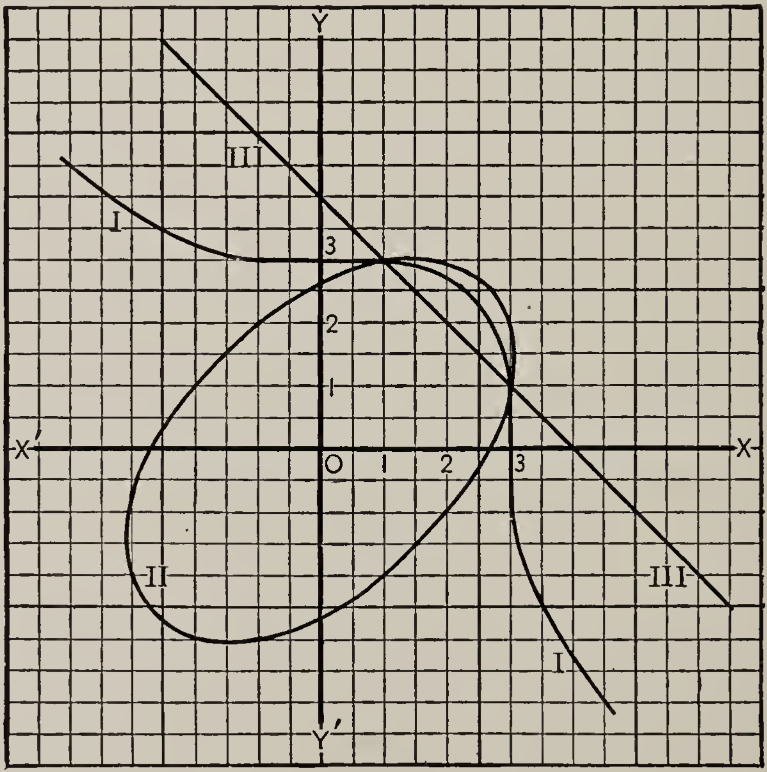
13. $x^3 - y^3 = 6x,$ $x - y = 3x.$

14. $x^3 + y^3 = 28,$ $x + y = 4.$

15. $x^2 - xy + y^2 = 7,$ $x^3 + y^3 = 28.$

16. $x + y = 4,$ $x^2 - xy + y^2 = 7.$

In the following figure I, II, and III are the graphs of $x^3 + y^3 = 28$, $x^2 - xy + y^2 = 7$, and $x + y = 4$ respectively. These equations are taken from the systems in Exercises 14, 15, and 16 which contain but three different equations paired in three ways. Since the two sets of roots for each is the same, we know that the three systems are equivalent. The equivalence of the three systems is also shown in the preceding figure.



MISCELLANEOUS EXERCISES

Solve by any method, pair results, and check each set of real roots:

(If any system cannot be solved algebraically by the methods previously given, solve it graphically.)

1. $$\begin{aligned}&2x^2+y^2=33,\\&x^2+2y^2=54.\end{aligned}$$

2. $$\begin{aligned}&3h^2-8k^2=40,\\&5h^2+k^2=81.\end{aligned}$$

3. $$\begin{aligned}&4R_1^2+3=9R_2^2,\\&12R_1^2+R_2^2=\tfrac{31}{9}.\end{aligned}$$

4. $$\begin{aligned}&xy+x=18,\\&xy+y=20.\end{aligned}$$

5. $$\begin{aligned}&x^2=y,\\&xy=8.\end{aligned}$$

6. $$\begin{aligned}&x-xy=5,\\&2y+xy=6.\end{aligned}$$

7. $$\begin{aligned}&x^3-y^3=19,\\&x-y=1.\end{aligned}$$

8. $$\begin{aligned}&x^3-y^3=19,\\&x^2+xy+y^2=19.\end{aligned}$$

9. $$\begin{aligned}&x^2+xy+y^2=19,\\&x-y=1.\end{aligned}$$

10. Show that the systems (7), (8), and (9) are equivalent by graphing the three equations of these exercises.

11. $$\begin{aligned}&3s^2-2t^2=0,\\&5s^2-3t^2=1.\end{aligned}$$

12. $$\begin{aligned}&4n^2+7m^2=9,\\&2n^2-\frac{9}{2}=m^2.\end{aligned}$$

13. $$\begin{aligned}&5W_1^2-6.8W_2^2=99.55,\\&W_1^2-W_2^2=20.\end{aligned}$$

14. $$\begin{aligned}&xy+2y^2=2,\\&3xy+5y^2=2.\end{aligned}$$

15. $$\begin{aligned}&x^2+2xy+2y^2=10,\\&3x^2-xy-y^2=51.\end{aligned}$$

16. $$\begin{aligned}&y^2+x=7,\\&x^2+y=11.\end{aligned}$$

17. $$\begin{aligned}&x^2+xy+y^2=7,\\&x^2+y^2=10.\end{aligned}$$

18. $$\begin{aligned}&x^2+xy+x=0,\\&x^2+xy+2x=0.\end{aligned}$$

19. $$\begin{aligned}&x^2+xy+y=0,\\&x^2+xy+x=0.\end{aligned}$$

20. $$\begin{aligned}&\frac{1}{x^2}+\frac{1}{y^2}=13,\\&\frac{1}{x}-\frac{1}{y}=1.\end{aligned}$$

21. $$\begin{aligned}&\frac{1}{x^3}-\frac{1}{y^3}=7,\\&\frac{1}{x}-\frac{1}{y}=1.\end{aligned}$$

22. $$\begin{aligned}&x^2-2xy+2y^2-y=0,\\&2x^2-3xy-y^2+2y=0.\end{aligned}$$

23. $$\begin{aligned} x^2 + z^2 &= 34, \\ x^2 + y^2 &= 25, \\ y^2 + z^2 &= 41. \end{aligned}$$

24. $$\begin{aligned} 3\,xy &= x^2y^2 - 88, \\ x - y &= 6. \end{aligned}$$

25. $$\begin{aligned} x^3 &= y^3 + 37, \\ x^2y &= xy^2 + 12. \end{aligned}$$

26. $$\begin{aligned} \frac{4}{x^2} - \frac{13}{xy} + \frac{9}{y^2} &= 9, \\ \frac{1}{xy} - \frac{1}{y^2} &= 3. \end{aligned}$$

27. $$\begin{aligned} x^2 &= 8\,x + 6\,y, \\ y^2 &= 6\,x + 8\,y. \end{aligned}$$

28. $$\begin{aligned} x^3 - y^3 &= 2\,x + y - 4, \\ x + 2\,y &= 4. \end{aligned}$$

29. $$\begin{aligned} xy &= c, \\ x + y &= a. \end{aligned}$$

30. $$\begin{aligned} x^{-2} - y^{-2} &= 6, \\ x^{-1} + y^{-1} &= 2. \end{aligned}$$

31. $$\begin{aligned} x - y &= 16, \\ x^{\frac{1}{2}} - y^{\frac{1}{2}} &= 2. \end{aligned}$$

32. $$\begin{aligned} \frac{1}{x-2} + \frac{1}{y-2} &= \frac{3}{4}, \\ \frac{1}{x} - \frac{1}{y} &= \frac{1}{12}. \end{aligned}$$

33. $$\begin{aligned} x^2 + 6\,x^{-2} &= 36\tfrac{1}{6}, \\ 3\,xy - x^2 &= 36. \end{aligned}$$

34. $$\begin{aligned} x - y\sqrt{x} &= 24, \\ x^2 + xy^2 &= 320. \end{aligned}$$

35. $$\begin{aligned} \frac{x-1}{y-1} &= 3, \\ \frac{y^2+y+1}{x^2-x+1} &= \frac{13}{43}. \end{aligned}$$

36. $$\begin{aligned} \frac{1}{x^2} + \frac{2}{xy} &= 16, \\ \frac{3}{x^2} - \frac{4}{xy} + \frac{2}{y^2} &= 6. \end{aligned}$$

37. $9\,x \div y = 18 = xy.$

38. $4\,x + \frac{1}{y} = 46 = \frac{26x}{5} - \frac{1}{y}.$

39. $$\begin{aligned} x^2 + y^2 - (y - x) &= 12, \\ x^2 - xy &= 0. \end{aligned}$$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problems.)

1. Find two numbers whose difference is 4 and the difference of whose squares is 88.

2. The sum of two numbers is 21 and the sum of their squares is 281. Find the numbers.

3. Find two numbers whose product is 192 and whose quotient is $\frac{3}{4}$.

4. The area of a right triangle is 150 square feet and its hypotenuse is 25 feet. Find the legs.

5. A rectangular field is 8 rods longer than it is wide and the area of the field is 8 acres. Find the length and the width.

6. The difference of the areas of two squares is 252 square feet, and the difference of their perimeters is 24 feet. Find a side of each square.

7. The area of a rectangular field is $3\frac{3}{5}$ acres and one diagonal is 60 rods. Find the perimeter of the field.

8. The perimeter of a rectangle is 112 feet and its area is 768 square feet. Find the length and the width.

9. A mean proportional between two numbers is $2\sqrt{14}$, and the sum of their squares is 113. Find the numbers.

10. The value of a certain fraction is $\frac{2}{3}$. If the fraction is squared and 44 is subtracted from both the numerator and the denominator of this result, the value of the fraction thus formed is $\frac{5}{14}$. Find the original fraction.

11. The base of a triangle is 6 inches longer than its altitude, and the area is $\frac{3}{2}$ square feet. Find the base and altitude of the triangle.

12. The volumes of two cubes differ by 1413 cubic inches and their edges differ by 3 inches. Find the edge of each.

13. The sum of the radii of two circles is 25 inches and the difference of their areas is $125\,\pi$ square inches. Find the radii.

14. The perimeter of a rectangle is $5\,C$ and its area is C^2. Find its dimensions.

15. The area of a right triangle is $8\,a^2 - 8\,b^2$ and its hypotenuse is $4\sqrt{2\,a^2 + 2\,b^2}$. Find the legs.

16. The perimeter of a right triangle is 56 feet and its area is 84 square feet. Find the legs and the hypotenuse.

17. If a 2-digit number be multiplied by the sum of its digits, the product is 324; and if three times the sum of its digits be added to the number, the result is expressed by the digits in reverse order. Find the number.

18. The yearly interest on a certain sum of money is \$42. If the sum were \$200 more and the interest 1% less, the annual income would be \$6 more. Find the principal and the rate.

19. A wheelman leaves A and travels north. At the same time a second wheelman leaves a point 3 miles east of A and travels east. One and one-third hours after starting, the shortest distance between them is 17 miles; and $3\frac{1}{3}$ hours later the distance is 53 miles. Find the rate of each.

20. The circumference of the fore wheel of a carriage is 1 foot less and that of its rear wheel 3 feet less than the circumferences of the corresponding wheels of a farm wagon. In going 1 mile the fore wheel of the carriage makes 40 revolutions more than its rear wheel, and the fore wheel of the wagon makes 88 more than its rear wheel. Find the circumferences of the carriage wheels.

21. A starts out from P to Q at the same time B leaves Q for P. When they meet, A has gone 40 miles more than B. A then finishes the journey to Q in 2 hours and B the journey to P in 8 hours. Find the rates of A and B, and the distance from P to Q.

22. A leaves P going to Q at the same time that B leaves Q on his way to P. From the time the two meet, it requires $6\frac{2}{3}$ hours for A to reach Q, and 15 hours for B to reach P. Find the rate of each, if the distance from P to Q is 300 miles.

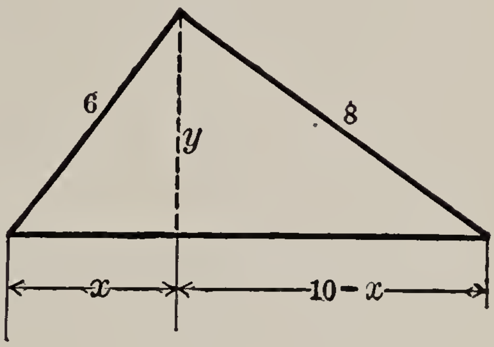
23. A man has a rectangular plot of ground whose area is 1250 square feet. Its length is twice its breadth. He wishes to divide the plot into a rectangular flower bed, surrounded by a path of uniform breadth, so that the bed and the path may have equal areas. Find the width of the path.

GEOMETRICAL PROBLEMS

1. The sides of a triangle are 6, 8, and 10. Find the altitude on the side 10.

HINT. From the adjacent figure we easily obtain the system:

$$\begin{cases} x^2 + y^2 = 36, \\ (10 - x)^2 + y^2 = 64. \end{cases}$$



2. The sides of a triangle are 8, 15, and 17. Find the altitude on the side 17 and the area of the triangle.

3. The sides of a triangle are 13, 20, and 21. Find the altitude on the side 20 and the area of the triangle.

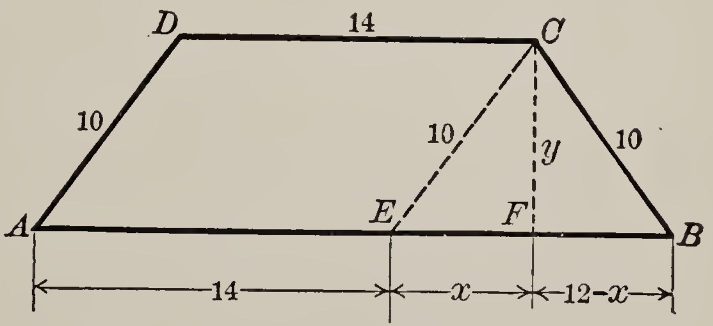
4. The sides of a triangle are 7, 15, and 20. Find the altitude on the side 7 and the area of the triangle.

5. The sides of a triangle are 10, 17, and 21. Find the altitude on the side 10 and the area of the triangle.

6. Find correct to two decimals the altitude on the side 16 of a triangle whose sides are 12, 16, and 18 respectively.

7. The parallel sides of a trapezoid are 14 and 26 respectively, and the two nonparallel sides are 10 each. Find the altitude of the trapezoid.

HINT. Let $ABCD$ be the trapezoid. Draw CE parallel to DA and CF perpendicular to AB. Then $EC = 10$, $AE = 14$, and $EB = 26 - 14$, or 12. If we let $EF = x$, FB must equal $12 - x$; then we can obtain the system of equations:



$$\begin{cases} x^2 + y^2 = 100, \\ (12 - x)^2 + y^2 = 100. \end{cases}$$

8. The two nonparallel sides of a trapezoid are 12 and 17 respectively, and the two bases are 5 and 13 respectively. Find the altitude of the trapezoid.

9. The bases of a trapezoid are 15 and 20 respectively, and the two nonparallel sides are 29 and 30. Find the altitude of the trapezoid and the area.

10. The sides of a trapezoid are 12, 20, 17, and 45. The sides 20 and 45 are the bases. Find the altitude and the area.

11. The sides of a trapezoid are 21, 27, 40, and 30. The sides 21 and 40 are parallel. Find the altitude and the area of the trapezoid.

12. The sides of a trapezoid are 23, 85, 100, and x. The sides 23 and 100 are the bases, and each is perpendicular to the side x. Find x and the area of the trapezoid.

13. The parallel sides of a trapezoid are 42 and 250. The other sides are 123 and 325. Find the altitude and the area of the trapezoid.

14. The area of a triangle is 1 square foot. The altitude on the first side is 16 inches. The second side is 14 inches longer than the third. Find the three sides.

CHAPTER XXXV

PROGRESSIONS

162. Definitions. In all fields of mathematics we frequently encounter groups of three or more numbers, selected according to some law and arranged in a definite order, whose relations to each other and to other numbers we wish to study.

The individual numbers or expressions are called *terms.*

In the following examples the law of formation and the order of the terms are so obvious that the student can write down many additional terms.

EXERCISES

Write three more terms in each of the following:

1. $1, 2, 3, 4, 5, \cdots.$

2. $2, 4, 6, 8, \cdots.$

3. $9, 8, 7, 6, \cdots.$

4. $-1, -3, -5, -7, \cdots.$

5. $1^2, 3^2, 5^2, \cdots.$

6. $\sqrt[3]{1}, \sqrt[3]{2}, \sqrt[3]{3}, \cdots.$

7. $2, 4, 8, 16, \cdots.$

8. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$

9. $1, \dfrac{1}{1}, \dfrac{1}{1\cdot 2}, \dfrac{1}{1\cdot 2\cdot 3}, \dfrac{1}{1\cdot 2\cdot 3\cdot 4}, \cdots.$

10. $1, 1+\sqrt{2}, 1+2\sqrt{2}, \cdots.$

There is an unlimited variety of such groups or successions of numbers. Only two simple types will be considered here.

163. Arithmetical progression. An **arithmetical progression** is a succession of terms in which each term after the first, minus the preceding one gives the same number.

This same number is called the **common difference** and may be any positive or negative number.

The numbers $3, 7, 11, 15, \cdots$ form an arithmetical progression, since any term after the first, minus the preceding one gives 4. Similarly, $12, 6, 0, -6, -12, \cdots$ is an arithmetical progression, since any term minus the preceding one gives the common difference -6. In like manner, $\frac{7}{2}, 5, 6\frac{1}{2}, \cdots$ is an arithmetical progression whose common difference is $1\frac{1}{2}$.

EXERCISES

From the following select the arithmetical progressions, and in each of them find the common difference:

1. $4, \frac{20}{3}, 9\frac{1}{3}, \cdots$
2. $10, 16\frac{1}{2}, 23, \cdots$.
3. $2, 4, 8, \cdots$.
4. $9, 12\frac{1}{2}, 15, \cdots$.
5. $25, 21, 17, \cdots$.
6. $18, 8, -2, \cdots$.
7. $5a+2, 3a+1, a, \cdots$.
8. $3x-5, 2x+8, x-7, \cdots$.
9. $\frac{9}{\sqrt{3}}, \frac{6}{\sqrt{3}}, \sqrt{3}, \cdots$.
10. $\frac{\sqrt{3}-2}{3}, \frac{2(\sqrt{3}-1)}{3}, \frac{1}{\sqrt{3}}, \cdots$.

164. The last or *n*th term of an arithmetical progression. If a denotes the first term and d the common difference, any arithmetical progression is represented by

$$a, a+d, a+2d, a+3d, a+4d, \text{etc.}$$

Here one observes that the coefficient of d in each term is one less than the number of the term. Hence the nth or general term is $a+(n-1)d$. If l denotes the nth term, we have

$$\boldsymbol{l = a + (n-1)d.} \qquad (A)$$

EXERCISES

1. Find the 12th term of the progression $1, 5, 9, 13, \cdots$.
2. Find the 23d term of the progression $-18, -15, -12, \cdots$.
3. Find the 15th term of the progression $13, 7, 1, -5, \cdots$.
4. Find the 19th term of the progression $a, 3a, 5a, \cdots$.
5. Find the 7th and 12th terms of the progression $\frac{7}{3}, 1\frac{2}{3}, 1, \cdots$.

6. Find the 5th and 20th terms of the progression $1, a+1, 1+2a, \cdots$

7. Find the 10th term of the progression $7\sqrt{2}, 5\sqrt{2}, 3\sqrt{2}, \cdots$.

8. Find the 9th term of the progression $\frac{\sqrt{3}+5}{2}, 2, \frac{3-\sqrt{3}}{2}, \cdots$.

9. Find the $(n-1)$st term of the progression $a, a+d, a+2d, \cdots$.

10. Find the $(n-2)$d term of the progression $a, a+d, a+2d, \cdots$.

11. Find the $(n-3)$d term of the progression $\sqrt{5}-1, 2\sqrt{5}-2, 3(\sqrt{5}-1), \cdots$.

12. Find the nth term of the progression $\frac{1}{n}, \frac{n-1}{n}, 2-\frac{3}{n}, \cdots$.

13. The first and second terms of an arithmetical progression are h and k respectively. Find the third term and the nth term.

14. The first and third terms of an arithmetical progression are h and k. Find the nth term.

15. A body falls 16 feet the first second, 48 the next, 80 the next, and so on. How far does it fall during the 10th second? during the nth second?

165. Arithmetical means. The arithmetical means between two numbers are numbers which form, with the two given ones as the first and the last terms, an arithmetical progression.

The insertion of one or more arithmetical means between two given numbers is performed as in the following:

Example: Insert three arithmetical means between 5 and 69.

Solution: $l = a + (n-1)d$.
There will be five terms in all.
Therefore $69 = 5 + (5-1)d$.
Solving, $d = 16$.
The required arithmetical progression is 5, 21, 37, 53, 69.

EXERCISES

1. Insert the arithmetical mean between 3 and 15.

2. Insert the arithmetical mean between h and $4k$.

3. Insert two arithmetical means between 2 and 17.

4. Insert two arithmetical means between a and b.

5. Insert three arithmetical means between -4 and 16.

6. Insert three arithmetical means between m and n.

7. Insert six arithmetical means between 3 and 45.

8. Insert nine arithmetical means between 3 and $\frac{14}{3}$.

9. Insert four arithmetical means between $-\sqrt{2}$ and $9\sqrt{2}$.

10. Insert five arithmetical means between $7x-3a$ and $13x+9a$.

11. Insert six arithmetical means between $\dfrac{5}{2\sqrt{5}}$ and $\dfrac{15\sqrt{5}}{2}$.

12. Insert two arithmetical means between $\dfrac{\sqrt{2}}{\sqrt{2}-1}$ and $\sqrt{2}(1-2\sqrt{2})$.

13. What is the arithmetical mean between any two numbers?

14. In going a distance of 1 mile an engine increased its speed uniformly from 20 miles per hour to 30 miles per hour. What was the mean or average velocity in miles per hour during that time? How long did it require to run the mile?

15. The velocity of a falling body increases uniformly. At the beginning of the third second its velocity is 64 feet per second, and at the end of the third second it is 96 feet per second. (*a*) What is its mean or average velocity in feet per second during the third second? (*b*) How many feet does it fall during the third second?

16. The velocity of a body falling from rest is 32 feet per second at the end of the first second. What is the mean or average velocity in feet per second during the first second? How many feet does the body fall during the first second? the second second?

17. Find the mean or average length of 25 lines whose lengths in inches are the first 25 even numbers.

18. Find the mean length of 17 lines whose lengths in inches are given by the consecutive odd numbers beginning with 11.

19. With the conditions of Problem 15 determine the average velocity per second of a body which has fallen for 10 seconds.

20. A certain distance is separated into 8 intervals, the lengths of which are in arithmetical progression. If the shortest interval is 1 inch and the longest 22 inches, find the others.

166. Sum of a series. The indicated sum of several terms of an arithmetical progression is called an **arithmetical series.** The formula for the sum of n terms of an arithmetical series may be obtained as follows:

$$S = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l. \quad (1)$$

Reversing the order of the terms in the second member of (1),

$$S = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a. \quad (2)$$

Adding (1) and (2),

$$2S = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l)$$
$$= n(a + l).$$

Therefore $$\boldsymbol{S = \frac{n}{2}(a + l).} \quad (B)$$

Substituting for l from (A), page 160,

$$S = \frac{n}{2}(a + [a + (n - 1)d]).$$

$$\boldsymbol{S = \frac{n}{2}[2a + (n - 1)d].} \quad (C)$$

EXAMPLE

Required the sum of the integers from 11 to 99 inclusive.

Solution: $n = 89$, $a = 11$, $l = 99$.

Substituting in (B), $S = \frac{n}{2}(a + l)$ gives $S = \frac{89(11 + 99)}{2} = 4895$.

Therefore the sum of the integers from 11 to 99 is 4895.

EXERCISES

1. Find the sum of 10 terms of the series $2 + 5 + 8 + \cdots$.

2. Find the sum of 18 terms of the series $10 + 8 + 6 + \cdots$.

3. Find the sum of 10 terms of the arithmetical progression $3, 4\frac{1}{3}, \frac{17}{3}, \cdots$.

4. Find the sum of 12 terms of the arithmetical progression $18, 14\frac{1}{2}, 11, \cdots$.

5. Find the sum of the first one hundred integers.

6. Find the sum of the first one hundred even numbers.

7. Find the sum of the first one hundred odd numbers.

8. Find the sum of the even numbers between 187 and 433.

9. Find the sum of the first n odd numbers.

10. Find the sum of the first n even numbers.

11. How many of the natural numbers beginning with 1 are required to make their sum 903?

HINT. Substitute in formula (C) preceding.

12. How many terms must constitute the series $5 + 9 + 13 + \cdots$ in order that it may amount to 275?

13. Beginning with 80 in the progression 78, 80, 82, how many terms are required to give a sum of 510? Explain.

14. The second term of an arithmetical progression is -7 and the seventh term is 18. Find the eleventh term.

15. Find the sum of t terms of the arithmetical progression $\frac{1}{t}, \frac{t-1}{t}, \cdots$.

16. If $l = 29$, $a = 2$, and $d = 3$, find n and s.

17. If $a = 3$, $d = 4$, and $s = 300$, find n and l.

18. If $d = -11$, $n = 13$, and $s = 0$, find a and l.

19. The first and second terms of an arithmetical progression are h and k respectively. Find the sum of n terms of the progression.

20. If $s = -33$, $a = 5x + 2$, and $n = 11$, find l and d.

21. If $s = 40\sqrt{2}$, $a = -5\sqrt{2}$, and $d = 2\sqrt{2}$, find n and l.

22. A clock strikes the hours but not the half hours. How many times does it strike in a day?

23. A car running 30 miles an hour is started up an incline, which decreases its velocity 2 feet a second. (*a*) In how many seconds will it stop? (*b*) How far will it go up the incline?

24. A car starts down a grade and moves 4 inches the first second, 12 inches the second second, 20 inches the third second, and so on. (*a*) How fast does it move in feet per second at the end of the twenty-first second? (*b*) How far has it moved in the twenty-one seconds?

25. An elastic ball falls from a height of 20 inches. On each rebound it comes to a point $\frac{1}{2}$ inch below the height reached the time before. How often will it drop before coming to rest? Find the total distance through which it has moved.

26. The digits of a 3-digit number are in arithmetical progression. The first digit is 2 and the number is $17\frac{1}{5}$ times the sum of its digits. Find the number.

27. A clerk received \$75 a month for the first year and a yearly increase of \$50 for the next ten years. Find his salary for the eleventh year and the total amount received.

28. Fifty dollars was deposited in a bank every first of March from February 28, 1893, to March 2, 1904. If the money drew simple interest at 3%, find the amount due the depositor on March 1, 1905.

29. Assuming that a ball is not retarded by the air, determine the number of seconds it will take to reach the ground if dropped from the top of the Washington Monument, which is 555 feet high. With what velocity will it strike the ground?

30. A ball thrown vertically upward rose to a height of 256 feet. In how many seconds did it begin to fall? With what velocity was it thrown?

31. A ball thrown vertically upward returned to the ground 7 seconds later. How high did it rise? With what velocity was it thrown?

32. A pyramid of billiard balls stands on an equilateral triangle, 10 balls on a side. How many balls are there in the bottom layer? in the whole pyramid?

33. A and B start from the same place at the same time and travel in the same direction. A travels 12 miles daily. B goes 7 miles the first day, $7\frac{1}{2}$ miles the second, 8 miles the third, and so on. When are they together?

34. A leaves P and travels south 2 miles the first day, 4 the second, 6 the third, and so on. Five days later B leaves P and travels south at the uniform rate of 28 miles a day. When are they together?

Note. In the earliest mathematical work known a problem is found which involves the idea of an arithmetical progression. In the papyrus of the Egyptian priest Ahmes, who lived nearly two thousand years before Christ, we read in essence, "Divide 40 loaves among 5 persons so that the numbers of loaves that they receive form an arithmetical progression, and so that the two who receive the least bread, together have one seventh as much as the others." From that time to this, the subject has been a favorite one with mathematical writers, and has been extended so widely that it would require several volumes to record all of the discoveries regarding the various kinds of series.

167. Geometrical progression. A **geometrical progression** is a succession of terms in which each term after the first, divided by the preceding one always gives the same number.

The constant quotient is called the **ratio.**

The numbers $2, 10, 50, 250, \cdots$, form a geometrical progression, since any term after the first, divided by the preceding one gives the same number 5. Similarly, the numbers $3, -3\sqrt{2}, 6, -6\sqrt{2}, \cdots$, form a geometrical progression, since any term after the first, divided by the preceding one gives the common ratio $-\sqrt{2}$.

EXERCISES

Determine which of the following are geometrical progressions and find in each case the corresponding ratio:

1. $2, 6, 18, \cdots$.
2. $15, 5, 1, \cdots$.
3. $18, -3, \frac{1}{2}, \cdots$.
4. $2, 4, 16, \cdots$.
5. $8, -4\sqrt{2}, 4, \cdots$.
6. $\sqrt{2}, \sqrt{\frac{1}{2}}, \frac{1}{4}\sqrt{2}, \cdots$.
7. $1, 3, 9, 81, \cdots$.
8. $\sqrt{\frac{2}{3}}, \sqrt{6}, \sqrt{54}, \cdots$.
9. $\frac{1}{\sqrt{8}}, -\frac{\sqrt{2}}{2}, 2, \cdots$.
10. $7a, 35a^2, 175a^3, \cdots$.
11. $8\sqrt{5}, -2\sqrt{5}, \sqrt{5}, \cdots$.
12. $5x^2, 10x^2y, 20x^2y^2, \cdots$.
13. $3xy^{\frac{1}{2}}, 12x^{\frac{1}{2}}y, 48y^{\frac{3}{2}}, \cdots$.

14. Find the condition under which a, b, and c form a geometrical progression.

168. The *n*th term of a geometrical progression. If a denotes the first term and r the ratio, any geometrical progression is represented by $a, ar, ar^2, ar^3, \cdots$. It is evident that the exponent of r in any term is one less than the number of the term. Therefore if t_n denotes the nth or general term of any geometrical progression,

$$t_n = ar^{n-1}. \qquad (A)$$

EXERCISES

1. Find the fifth term of 4, 12, 36.

Solution: Here $a = 4$, $r = 3$, $n - 1 = 4$.

Substituting these values in the formula $t_n = ar^{n-1}$,

$$t_5 = 4 \cdot 3^4 = 324.$$

2. Find the tenth term of $3, 6, 12, \cdots$.

3. Find the eighth term of $2, 3, \frac{9}{2}, \cdots$.

4. Find the twelfth term of $5, -10, 20, \cdots$.

5. Find t_6 of the geometrical progression \$100, \$106, \$112.36, ⋯.

6. Find t_9 of the geometrical progression $18, -6, +2, \cdots$.

7. Find t_{10} of the geometrical progression $12a, 9a, \frac{27a}{4}, \ldots$.

8. Find t_7 of the geometrical progression $\frac{-2c}{3}, -1, \frac{-3}{2c}, \ldots$.

9. Find t_7 of the geometrical progression $4\sqrt{2}, 4, 2\sqrt{2}$.

10. Find t_6 of the geometrical progression $\frac{1}{2\sqrt{2}}, \frac{1}{6}, \frac{\sqrt{2}}{18}$.

11. Find t_8 of the geometrical progression $\frac{3}{2\sqrt{2}}, 1, \frac{2\sqrt{2}}{3}$.

12. The nth term of a geometrical progression is ar^{n-1}. What is the $(n-1)$st term? the $(n-2)$d? the $(n-3)$d? the $(n+1)$st? the $(n+2)$d?

13. The first and second terms of a geometrical progression are h and k respectively. Find the next two terms.

169. Geometrical means. **Geometrical means** between two numbers are numbers which form, with the two given ones as the first and the last terms, a geometrical progression.

EXERCISES

1. Insert two geometrical means between 9 and 72.

Solution: There are four terms in the geometrical progression, $a = 9$, $n = 4$, and $t_n = t_4 = 72$.

Substituting these values in $t_n = ar^{n-1}$,

$$72 = 9r^3.$$

Whence $r = 2$.

The required geometrical progression is 9, 18, 36, 72.

2. Insert two geometrical means between 6 and 48.

3. Insert three geometrical means between 6 and 486.

4. Insert one geometrical mean between 4 and 9.

5. Insert one geometrical mean between a^{10} and a^{20}.

6. Insert three geometrical means between -144 and -9.

7. The fifth term of a geometrical progression is 32, the ninth term is 512. Find the eleventh term.

8. The second term of a geometrical progression is $3\sqrt{2}$, the fifth term is $\frac{3}{16}$. Find the first term and the ratio.

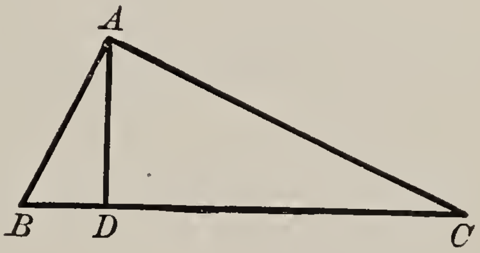
9. Show that the geometrical means between h and k are $\pm\sqrt{hk}$.

10. The first and fourth terms of a geometrical progression are h and k. Find the second and third terms.

11. Insert three geometrical means between a and c.

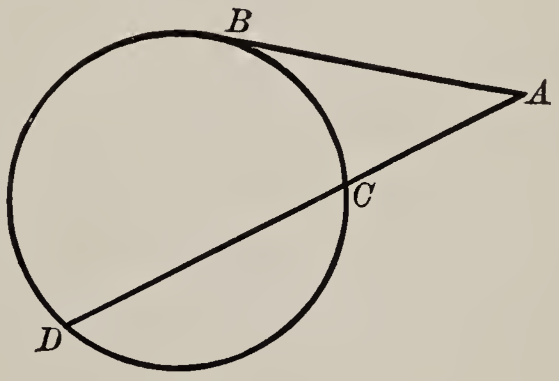
12. The sum of the first and third terms of a geometrical progression is 13 and the second term is 6. Find each term.

13. In the adjacent figure ABC is a right triangle and AD is perpendicular to the hypotenuse BC. Under these conditions the length of AD is *always* a geometric mean between the lengths of BD and DC.



(*a*) If $BD = 4$ and $DC = 9$, find AD.
(*b*) If $BC = 26$ and $AD = 12$, find BD and DC.

14. In the adjacent figure AB touches and AD cuts the circle. Under such conditions the length of AB is *always* a geometric mean between the lengths of AC and AD.



(*a*) If $AD = 16$ and $AC = 9$, find AB.
(*b*) If $DC = 24$ and $AB = 16$, find AC and AD.

170. Geometrical series. Let S_n denote the indicated sum of n terms of a geometrical progression. This indicated sum is called a **geometrical series.** Obtaining in its simplest form the expression for this sum is often called *finding the sum of the series.*

The expression for the sum is derived as follows:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}. \quad (1)$$

$$(1)\cdot r, \quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \quad (2)$$

The terms ar, ar^2, etc., up to ar^{n-1} in the right member of (1), occur in the right member of (2). Hence if (2) be subtracted from (1), all these terms vanish, leaving only a and ar^n.

(1) – (2), $S_n - rS_n = a - ar^n$.

Whence $S_n(1-r) = a - ar^n$,

and
$$\mathbf{S_n = \frac{a - ar^n}{1 - r}}. \quad (B)$$

EXERCISES

1. Find the sum of the first ten terms of $5, -10, 20, \cdots$.

Solution: $S_n = \dfrac{a - ar^n}{1 - r}$.

By the conditions, $a = 5$, $r = -2$, and $n = 10$.

Substituting, $S_{10} = \dfrac{5 - 5(-2)^{10}}{1 - (-2)} = 1705$.

2. Find the sum of $1, 5, 25, \cdots$ to seven terms.

3. Find S_7 for the progression $-2, 4, -8, \cdots$.

4. Find S_8 for the progression $50, 10, 2, \cdots$.

5. Find S_9 for the progression $180, -90, 45, \cdots$.

6. Find S_5 for the progression $\frac{2}{5}, 1, \frac{5}{2}, \cdots$.

7. Find S_7 for the progression $c^3, c^5, c^7, \cdots$.

8. Find S_5 for the progression $3\sqrt{2}, 6, 6\sqrt{2}, \cdots$.

9. Find S_6 for the progression $81, -27\sqrt{3}, 27, \cdots$.

10. Find S_n for the progression $3, 15, 75, \cdots$.

11. Find S_{n-2} for the progression $2x, 4x^4, 8x^7, \cdots$.

12. Show that for a geometrical progression $S_n = \dfrac{a - rl}{1 - r}$.

13. What will \$100 amount to in three years, interest 4%, compounded annually? compounded semiannually?

14. A rubber ball falls from a height of 40 inches and on each rebound rises 40% of the previous height. How far does it fall on its sixth descent? Through what distance has it moved at the end of the sixth descent?

15. A vessel containing wine was emptied of one third of its contents and then filled with water. This was done six times. What portion of the original contents was then in the vessel?

16. At each stroke an air pump withdraws 40 cubic inches of the contents of a bell jar whose capacity is 400 cubic inches. After every stroke the air remaining in the jar expands and completely fills it. What portion of the original quantity of air remains in the jar at the end of the tenth stroke?

171. Infinite geometrical series. If the number of terms of a geometrical series is unlimited, it is called an infinite geometrical series.

In the progression 2, 4, 8, $\cdots$ the ratio is positive and greater than 1, and each term is greater than the term preceding it. Such a progression is said to be increasing. Obviously the sum of an unlimited number of terms of an increasing geometrical progression is unlimited. In other words, by taking enough terms the sum can be made as large as we please.

In the progression $3, \frac{3}{2}, \frac{3}{4}, \cdots$ the ratio is positive and less than 1, and each term is less than the term preceding it. Such a progression is said to be decreasing. Though the number of terms of such a geometrical progression be unlimited, the sum is limited; that is, the sum of as many terms as we choose to take is always *less* than some definite number. The sum of the first 3 terms of the series $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is 7; of 4 terms is $7\frac{1}{2}$; of 5 terms is $7\frac{3}{4}$; of 6 terms is $7\frac{7}{8}$; of 7 terms is $7\frac{15}{16}$. Here, for any number of terms, the sum is always less than 8.

The formula
$$S_n = \frac{a - ar^n}{1 - r} \qquad (1)$$

may be written
$$S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \qquad (2)$$

For the series $3+\frac{3}{2}+\frac{3}{4}+\cdots$,

$$S_n=\frac{3}{1-\frac{1}{2}}-\frac{3(\frac{1}{2})^n}{1-\frac{1}{2}}. \tag{3}$$

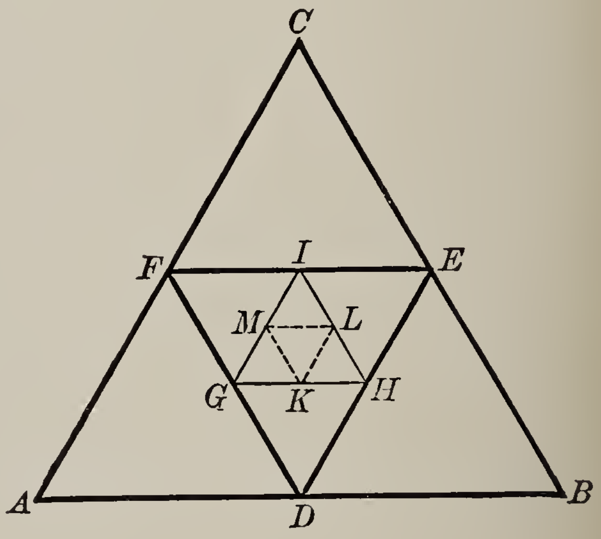
Now $(\frac{1}{2})^2=\frac{1}{4}$, $(\frac{1}{2})^3=\frac{1}{8}$, $(\frac{1}{2})^4=\frac{1}{16}$, $(\frac{1}{2})^5=\frac{1}{32}$. Consequently $(\frac{1}{2})^n$ becomes very small if n is taken very great. Therefore $3(\frac{1}{2})^n$, the numerator of the last fraction in (3), decreases and approaches zero as n increases without limit. And hence as the denominator of the fraction remains $\frac{1}{2}$ while the numerator approaches zero, the value of the fraction decreases and approaches zero as n increases. Then if S_∞ denotes S_n, where n has increased without limit, we may write

$$S_\infty \text{ approaches } \frac{3}{1-\frac{1}{2}} \text{ or } 6.$$

This means that, though n be very large, the sum of the series $3+\frac{3}{2}+\frac{3}{4}+\cdots$ is always slightly less than 6.

The following is a geometrical illustration of the preceding series:

In the adjacent figure triangles ABC, DEF, GHI, etc., are equilateral. DEF is formed by joining the middle points of the sides of ABC, etc. Imagine this process continued until an unlimited number of triangles is so formed. Now FE is $\frac{1}{2}$ of AB, GH is $\frac{1}{2}$ of FE, ML is $\frac{1}{2}$ of GH, etc. Therefore, if $AB=1$, $FE=\frac{1}{2}$, $GH=\frac{1}{4}$, $ML=\frac{1}{8}$, etc. Hence the perimeter of ABC is 3; of DEF, $\frac{3}{2}$; of GHI, $\frac{3}{4}$; etc. Thus the perimeters of the successive triangles form the progression $3, \frac{3}{2}, \frac{3}{4}, \cdots$, the limit of whose sum was found to be 6.



In the general case, if r is numerically less than 1, the numerical value of fraction $\frac{ar^n}{1-r}$ approaches zero as n increases without limit. Under such conditions the formula

$$S_r=\frac{a}{1-r}-\frac{ar^n}{1-r} \text{ becomes } S_\infty=\frac{a}{1-r}.$$

This means that for r numerically less than 1, S_n approaches $\frac{a}{1-r}$; but for any definite value of n it is always numerically less than this number.

Hence whenever we speak of the sum of such a series we mean the *limit* which the sum approaches as n increases indefinitely.

EXERCISES

Find the number which the sum of the first n terms of each of the following approaches as n increases without limit:

1. $3, 1, \frac{1}{3}, \cdots$.

Solution: $$S_\infty = \frac{a}{1-r}.$$

Substituting, $$S_\infty = \frac{3}{1-\frac{1}{3}} = 4\tfrac{1}{2}.$$

2. $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

3. $2, -1, \frac{1}{2}, \cdots$.

4. $3, -1, \frac{1}{3}, \cdots$.

5. $2, \sqrt{2}, 1, \cdots$.

6. $5a, \frac{5a}{4}, \frac{5a}{16}, \cdots$.

7. $1, x, x^2, \cdots, (x < 1)$.

8. $3, \sqrt{3}, 1, \cdots$.

9. $1, \frac{1}{x}, \frac{1}{x^2}, \cdots, (x > 1)$.

10. .515151. Hint. $.515151 = \frac{51}{100} + \frac{51}{10000} + \frac{51}{1000000} + \cdots$.

11. .666 ⋯.

12. .272727 ⋯.

13. .3939 ⋯.

14. 25.3636 ⋯.

15. .72121 ⋯.

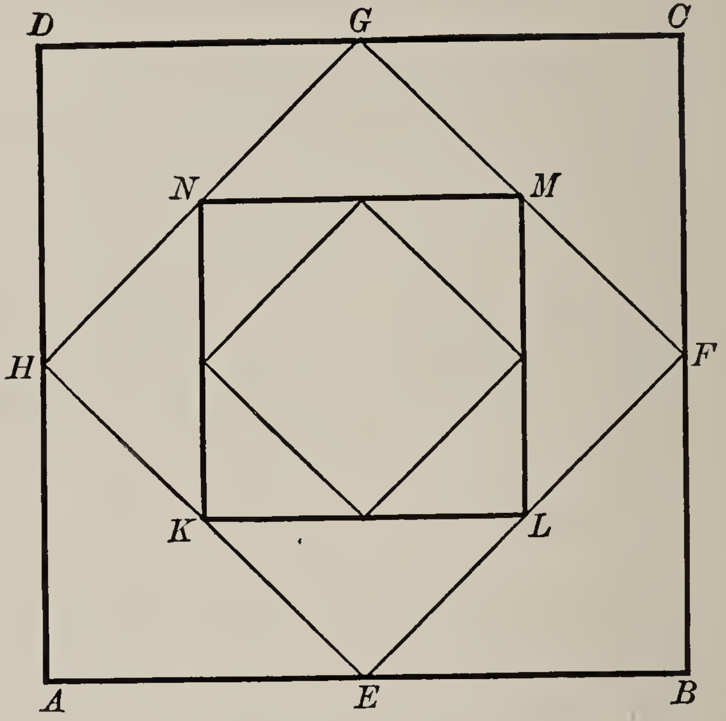
16. .3091091 ⋯.

17. A flywheel whose perimeter is 5 feet makes 80 revolutions per second. If it makes 99% as many revolutions each second thereafter as it did the preceding second, how far will a point on its rim have moved by the time it is about to stop?

18. The area of the triangle ABC (page 420) is $\frac{9}{4}\sqrt{3}$; of triangle DEF, $\frac{9}{16}\sqrt{3}$; of triangle GHI, $\frac{9}{64}\sqrt{3}$, etc. Find the sum of the areas of all the triangles drawn as there supposed.

19. The square $EFHG$ is formed by joining the middle points of the adjacent sides of the square $ABCD$ on page 422.

If an unlimited number of squares is so formed, the perimeters of the squares will form a geometrical progression the first three terms of which may be obtained as follows: Let $AB = 2$; then $EB = BF = 1$. EF can then be found from right triangle EBF; EL is $\frac{1}{2}$ of EF, and $EK = EL$. Then KL can be found from the right triangle KEL. The perimeters of the first three squares can then be found.



(*a*) Show that the limit of the sum of the perimeters of all of the squares is

$$16 + 8\sqrt{2}.$$

(*b*) Show that the limit of the sum of the areas of all the squares is 8 square units.

20. A loan of S dollars is to be repaid in four equal annual payments of p dollars each. Find p if money is worth $r\%$.

Solution: The sum due at beginning of second year

$$= S(1 + r) - p. \qquad (1)$$

The sum due at beginning of third year

$$= [S(1 + r) - p](1 + r) - p. \qquad (2)$$

The sum due at beginning of fourth year

$$= \{[S(1 + r) - p](1 + r) - p\}(1 + r) - p. \qquad (3)$$

The sum due at beginning of fifth year

$$= [\{[S(1 + r) - p](1 + r) - p\}(1 + r) - p](1 + r) - p. \qquad (4)$$

By the conditions of the problem, (4) = 0, for all the debt has then been paid. Setting (4) equal to zero and simplifying,

$$S(1 + r)^4 - p(1 + r)^3 - p(1 + r)^2 - p(1 + r) - p = 0. \qquad (5)$$

Solving (5) for p,

$$p = \frac{S(1 + r)^4}{(1 + r)^3 + (1 + r)^2 + (1 + r) + 1}. \qquad (6)$$

But the denominator in (6) is a geometrical series whose sum by formula (*B*), page 418, is $\frac{(1 + r)^4 - 1}{r}$.

Substituting this last in (6), $p = \frac{Sr(1+r)^4}{(1+r)^4 - 1}$. (7)

In the general case, if we have n annual payments, the exponent 4 in (7) would be replaced by n, and then $p = \frac{Sr(1+r)^n}{(1+r)^n - 1}$.

21. A loan of \$1000 is to be repaid in three equal annual payments, interest at 5%. Find the payment.

22. A loan of \$5000 bearing interest at 6% is to be repaid in five equal annual payments. Find the payment.

Note. In the study of geometrical progressions we have seen that the sum of the infinite series $1 + x^2 + x^3 + x^4 + \cdots$ is a definite number when x has any value less than one. But it has no finite value when x is equal to or greater than one; that is, we have an expression which we cannot use arithmetically unless x has a properly chosen value. If we were studying some problem which involved such a series, it would be a matter of the most vital importance to know whether the values of x under discussion were such as to make the series meaningless.

This question of distinguishing between expressions the sum of whose terms approach a limit or converge, and those which do not, has an interesting history. Newton and his followers in the seventeenth century dealt with infinite series and always assumed that they converged, as, in fact, most of them did. But as more complicated series came into use it became more difficult to tell from inspection whether they meant anything or not for a given value of the variable.

It was not until the beginning of the nineteenth century that Gauss, Abel, and Cauchy, in Germany, Norway, and France, respectively began to study this subject effectively, and to devise far-reaching tests to determine the values of x for which certain series converge to a finite limit. It is said that on hearing a discussion by Cauchy in regard to series which do not always converge, the astronomer La Place became greatly alarmed lest he had made use of some such series in his great work on Celestial Mechanics. He hurried home and denied himself to all distractions until he had examined every series in his book. To his intense satisfaction they all converged. In fact, it has often been observed that a genius can safely take chances in the use of delicate processes, which seem very foolish and unsafe to a man of ordinary ability.

CHAPTER XXXVI

LIMITS AND INFINITY

172. Limits. The numerical value V of the recurring decimal $.666\cdots$ is a variable depending on the number of 6's annexed on the right. Every 6 thus repeated increases V, and the number of 6's which may be so repeated is unlimited. Still V always remains less than $\frac{2}{3}$, though constantly approaching nearer and nearer to that value. Here the fraction $\frac{2}{3}$ is called the **limit** of the variable V.

173. Definition of a limit. If a variable V takes on successively a series of values that approach nearer and nearer to a fixed number L in such a manner that the numerical value of $V - L$ becomes and remains as small as we please, then V is said to approach the *limit* L.

This may be written limit of $V = L$.

The symbol $\doteq$ gives us the equivalent notation $V \doteq L$, which is read V *approaches* L *as a limit.*

174. Infinity. If a variable n takes on in succession all the values $1, 2, 3, 4, \cdots$, we can conceive of no final value for n, since the system of natural numbers is unlimited. Here we may say n *increases without limit*, or n becomes **infinite.**

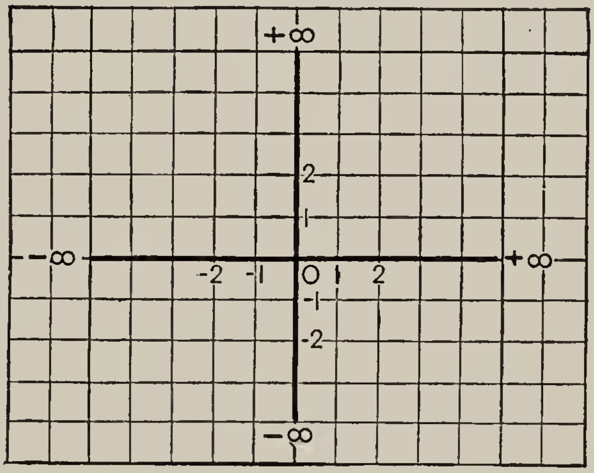
175. Definition of the term "infinite." If a variable n becomes and remains greater than any positive number k, however great, we say n *increases without limit*, or n becomes *infinite.*

The usual symbol for a variable which has become infinite is the sign ∞, read **infinity.**

Infinity is not a number in the sense in which 2, $\sqrt{6}$, and -7 are numbers. It is greater than any number. For present purposes it must be regarded as a manner of speech rather than as a number that can be added, subtracted, multiplied, or divided, as ordinary numbers are. In fact, we cannot operate with the symbol ∞ as we can with numbers.

Note. Some idea of the reason why we cannot regard ∞ as a number, and operate with it as we do with ordinary numbers, may be seen if we consider all even numbers, 2, 4, 6, 8, 10, $\cdots$. Evidently they may be continued as far as we wish, but the number of them all cannot be expressed by any integer, for it is greater than any number; that is, it is infinite. But the number of all integers, both odd and even, was also called infinite, and we symbolize both infinities by the same sign, ∞. We have, then, two infinities which are equal, or at least they are represented by the same symbol, but one contains the other. This is contrary to the axiom which we always assume for finite numbers; namely, that the whole is greater than any of its parts. Surely it is not strange that we cannot operate freely with a symbol which violates this fundamental principle.

We may have a negative infinity as well as a positive one. In order to indicate the range of values which both x and y may take in graphical work, the axes are often marked as in the adjacent figure.



A *constant* number, however large, is never spoken of as infinite.

If the variable n in $\frac{1}{n}$ takes on in succession the values 1, 2, 3, 4, $\cdots$, no final value of $\frac{1}{n}$ can be imagined. But as n increases without limit $\frac{1}{n}$ becomes very small and approaches nearer and nearer to zero without actually becoming zero.

In general, if a in the fraction $\frac{a}{n}$ is any constant not zero, and n a variable increasing without limit, $\frac{a}{n}$ approaches zero as a limit. Unfortunately, in elementary mathematics there is not in general use a symbol for a variable whose limit is zero, though such a symbol would be a great convenience.

The student will frequently meet the statement $\frac{a}{\infty}$ equals zero. This statement is, of course, meaningless until it has been defined, but it may properly be regarded as a way of saying that $\frac{a}{n}$ approaches zero as a limit when n is indefinitely increased.

176. Interpretation of $\frac{a}{0}$. Division by zero is excluded from mathematics for two reasons: (*a*) It is never necessary. (*b*) It would give rise to endless ambiguities and difficulties.

Results of the form $\frac{a}{0}$, where a is a constant not zero, frequently arise. According to the rules of computation, however, such an expression has no meaning. Though it is true that $\frac{a}{0}$ is not a definite number, results of this form may sometimes admit of interpretation.

As an illustration of this, consider the following

EXAMPLE

Solve by determinants the system

$$\begin{cases} x - 2y = 1, & (1) \\ \frac{1}{2}x - y = 2. & (2) \end{cases}$$

Solution:

$$x = \frac{\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ \frac{1}{2} & -1 \end{vmatrix}} = \frac{-1+4}{1-1} = \frac{3}{0},$$

and

$$y = \frac{\begin{vmatrix} 1 & 1 \\ \frac{1}{2} & 2 \end{vmatrix}}{0} = \frac{2-\frac{1}{2}}{0} = \frac{1.5}{0}.$$

The graphs of (1) and (2) are parallel lines. For such lines there is no point of intersection and consequently the system has no set of roots. Now as the results for x and y are of the form $\frac{a}{0}$, the attempt at solution by determinants fails.

Therefore the interpretation of these results is that no set of roots exists for the system (1), (2).

In general, if for any system of linear equations the results obtained are of the form $\frac{a}{0}$, the system has no set of roots; that is, the system is inconsistent.

If the student meets the statement $\frac{a}{0} = \infty$, he should regard it as a loose use of the statement that $\frac{a}{n}$ becomes infinite as n approaches 0.

177. Interpretation of $\frac{0}{0}$. The fraction $\frac{x-2}{x^2-4}$ becomes $\frac{0}{0}$ when $x = 2$. For any value of x other than the critical value 2, the fraction equals a definite number. Usually we are concerned with the limit of such expressions as the variable approaches a critical value. The limit for the fraction $\frac{x-2}{x^2-4}$ is easily found. We assign to x successively the values 1.9, 1.99, 1.999, 1.9999, etc. The corresponding values of the fractions are $\frac{10}{39}$, $\frac{100}{399}$, $\frac{1000}{3999}$, $\frac{10000}{39999}$, etc. Obviously these numbers approach the limit $\frac{1}{4}$.

We may arrive at this result more easily as follows: For all values of x except 2 the terms of the fraction $\frac{x-2}{x^2-4}$ may be divided by $x - 2$, obtaining $\frac{1}{x+2}$. This result is true, however little x may differ from 2. Now if, without giving x the value 2, we make it approach 2 as a limit, $\frac{1}{x+2}$ will approach $\frac{1}{4}$ as a limit, and this is the limit of the original fraction $\frac{x-2}{x^2-4}$ as well.

By either of the preceding methods it can be shown that $\frac{x^2-9}{x-3}$, which becomes $\frac{0}{0}$ for $x = 3$, has 6 as its limit. These two fractions are simple illustrations of the important fact that the symbol $\frac{0}{0}$ is not a definite number. The truth of this can be seen more clearly from a study of the graph on the following page.

Let $$y = \frac{x^2-9}{x-3}. \tag{1}$$

Then $$y(x-3) = (x+3)(x-3). \tag{2}$$

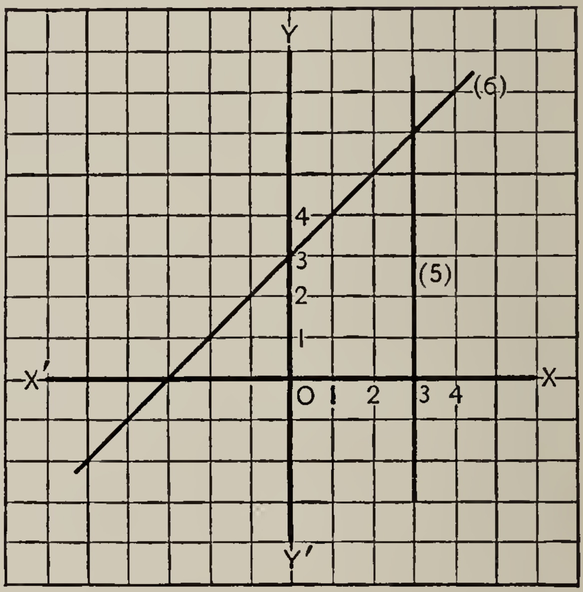
$$y(x-3) - (x+3)(x-3) = 0. \tag{3}$$

$$(y-x-3)(x-3) = 0. \tag{4}$$

Therefore $$x - 3 = 0, \tag{5}$$

and $$y - x - 3 = 0. \tag{6}$$

The graphs of (5) and (6) are given in the following figure. To understand what follows, it must be remembered that y and the fraction $\frac{x^2-9}{x-3}$ are *identical*, and that the graphs (5) and (6) are the complete graph of the equation (1).



For every value of x except 3 there is always one value of y, and that value is the y-distance of some point on line (6). For $x = 3$, however, the value of y is the y-distance of *any* point on line (5). Hence the fraction $\frac{x^2-9}{x-3}$ is indeterminate for $x = 3$.

It is worth noting that the limiting value of the fraction $\frac{x^2-9}{x-3}$ is seen from the graph to be 6, the y-value of the point of intersection of lines (5) and (6).

Note. The study of the limiting value of the ratio of two functions which for certain values of the variable takes on the indeterminate form 0/0 was undertaken by the Frenchman, L'Hospital, in 1696, and was carried further by John Bernouilli a few years later. A complete comprehension of the difficulties which surround this subject has been very slowly gained by mathematical writers, and even to-day it is possible to find books in which grave errors are made regarding the meaning of these expressions.

The questions involved are closely related to those regarding the nature of the infinite in mathematics. The penetration of this mystery is one of the great achievements of the latter half of the nineteenth century, and to-day well-informed mathematicians have as clear and satisfactory ideas about infinite numbers as they do about ordinary integers.

As a final illustration that $\frac{0}{0}$ may have any value, consider the following

EXAMPLE

Solve by determinants the system $\begin{cases} x - y = 1, & (1) \\ -3x + 3y = -3. & (2) \end{cases}$

Solution: $x = \dfrac{\begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix}} = \dfrac{3-3}{3-3} = \dfrac{0}{0}.$

$y = \dfrac{\begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix}}{0} = \dfrac{-3+3}{0} = \dfrac{0}{0}.$

Now the graphs of equations (1) and (2) are coincident lines. Therefore any set of values of x and y which satisfies (1) will also satisfy (2), a condition indicated by the indeterminate result $\frac{0}{0}$ for the unknowns.

In general, if the solution of a system of linear equations in two or more unknowns gives $\frac{0}{0}$ as values of the unknowns, the system has an infinite number of sets of roots; that is, the system is indeterminate.

The symbol $\frac{0}{0}$ then is a symbol of indetermination.

EXERCISES

Solve by determinants and interpret results:

1. $x - y = 1,$
   $2y - 2x = -2.$

2. $x - y = 0,$
   $y - x = 3.$

3. $x + y + z = 0,$
   $x - 2y + 3z = 1,$
   $2x - y + 4z = 1.$

4. $x + y + z = 1,$
   $x - y - 2z = 2,$
   $0x + 0y + 0z = 0.$

5. From the results obtained in Exercise 4 what conclusion is warranted regarding the number of sets of roots belonging to a system of *two* linear equations in *three* variables?

6. $x + y + z = 2,\ 0x + 0y + 0z = 0,\ 0x + 0y + 0z = 0.$

7. What do the results obtained in Exercise 6 show in regard to the number of sets of roots belonging to *one* equation in *three* variables?

What limit does each of the following expressions approach as n becomes ∞?

8. $\frac{1}{n}$. **10.** $\frac{3}{n}$. **12.** $\frac{n}{n+2}$. **14.** $\frac{4}{\frac{1}{n}}$.

9. $\frac{2}{n}$. **11.** $\frac{n}{n+1}$. **13.** $\frac{n+1}{n}$.

15. $\frac{n(n+1)}{n^2}$. **16.** $\frac{n(n+1)(n+2)}{n^3}$.

What limit does each of the following expressions approach as $n \doteq 0$?

17. $\frac{1}{n}$. **18.** $\frac{4}{\frac{1}{n}}$. **19.** $\frac{6}{\frac{1}{n}}$. **20.** $2n^2$. **21.** $\frac{n}{n^2}$.

Find the limit of:

22. $\frac{1-x}{1-x^2}$ as $x \doteq 1$. **23.** $\frac{x^2-5x+6}{x^2-4}$ as $x \doteq 2$.

24. $\frac{x-2}{x^3-8}$ as $x \doteq 2$.

CHAPTER XXXVII

LOGARITHMS

178. Introduction. Logarithms were invented to shorten the work of extended numerical computations which involve one or more operations of multiplication, division, involution, and evolution. They have decreased the labor of computing to such an extent that many calculations which would require hours without logarithms can be performed by their aid in one tenth of that time.

A logarithm is an exponent. A table of common logarithms is a table of exponents of the number 10. The greater portion of these exponents are approximate values of irrational numbers. It follows, then, that computation by means of logarithms gives only approximate results. Tables exist, however, in which each logarithm is given to twenty or more decimals; hence practically any desired degree of accuracy can be obtained by using the proper table.

It can be proved that the laws given on page 352, governing the use of rational exponents, hold for irrational exponents. In the work on logarithms this fact will be assumed.

179. Graphical explanation of logarithms. The theory of computation by logarithms is simple, yet considerable time is needed to master its practical details. These details and the fact that a logarithm is an exponent will be grasped more readily if the student gets from a graph a first view of the meaning and use of logarithms. For this we shall construct the graph of the logarithmic or exponential equation,

$$N = 10^L.$$

In this equation N represents any positive number, and L, the exponent of 10, is its common logarithm.

It will be more convenient to assign values to L and compute the corresponding values of N, than to use the reverse process. Moreover, we shall restrict L to values from $+1$ to -1 inclusive, and to such fractional values that N can be obtained by the use of square root.

First, $10^1 = 10$, $10^0 = 1$, and $10^{-1} = .1$,

and $10^{\frac{1}{2}} = \sqrt{10} = 3.16227 +$.

Also $10^{\frac{1}{4}} = (10^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\sqrt{10}} = \sqrt{3.16227} = 1.778 +$.

Similarly, $10^{\frac{1}{8}} = (10^{\frac{1}{4}})^{\frac{1}{2}} = \sqrt[2]{\sqrt[4]{10}} = \sqrt[2]{1.778} = 1.33 +$.

Now $10^{\frac{3}{4}} = (10^{\frac{1}{2}})(10^{\frac{1}{4}}) = (3.16227)(1.778) = 5.62 +$.

Again, $10^{\frac{3}{8}} = (10^{\frac{1}{4}})(10^{\frac{1}{8}}) = (1.778)(1.33) = 2.37 +$.

In like manner,

$$10^{\frac{5}{8}} = (10^{\frac{3}{8}})(10^{\frac{1}{4}}) = 4.21 +.$$

Lastly, $10^{\frac{7}{8}} = (10^{\frac{1}{8}})(10^{\frac{3}{4}}) = 7.49 +$.

Tabulating the values of N and L just obtained, gives

L	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
N	1	1.33	1.78	2.37	3.16	4.21	5.62	7.49	10

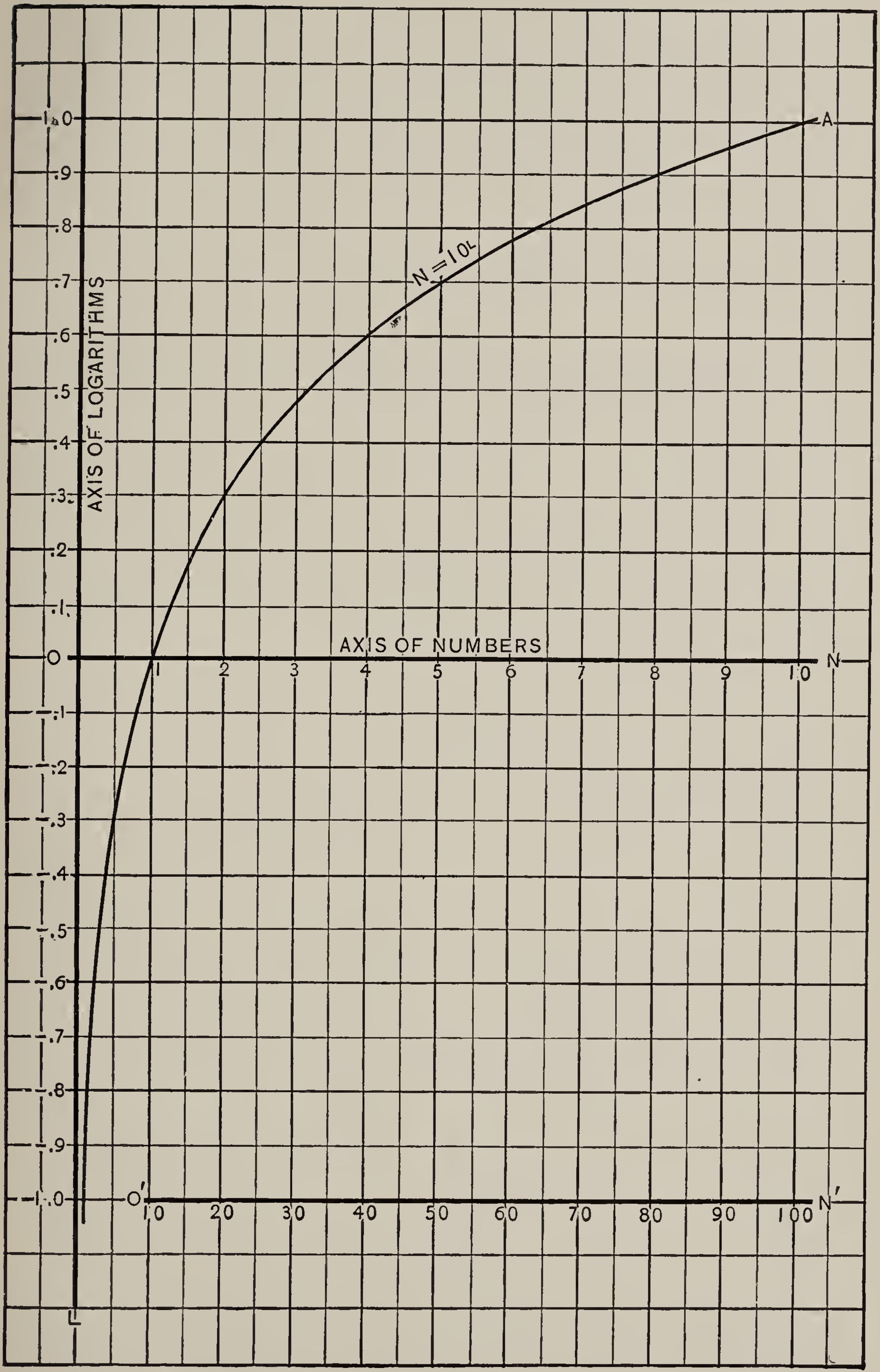
Since $$10^{-L} = \frac{1}{10^L},$$

the value of $$10^{-\frac{7}{8}} = \frac{1}{10^{\frac{7}{8}}} = \frac{10^{\frac{1}{8}}}{10^{\frac{7}{8}} \cdot 10^{\frac{1}{8}}} = \frac{10^{\frac{1}{8}}}{10} = .133.$$

Similarly, $$10^{-\frac{3}{4}} = \frac{1}{10^{\frac{3}{4}}} = \frac{10^{\frac{1}{4}}}{10} = .1778.$$

In this manner we obtain from the preceding table the following one for negative values of L between 0 and -1.

L	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{1}{2}$	$-\frac{5}{8}$	$-\frac{3}{4}$	$-\frac{7}{8}$	-1
N	.749	.562	.421	.316	.237	.178	.133	.1


AXIS OF LOGARITHMS
AXIS OF NUMBERS
$N = 10^L$
A
N
N′
O′
L
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
−.1
−.2
−.3
−.4
−.5
−.6
−.7
−.8
−.9
−1.0
1
2
3
4
5
6
7
8
9
10
10
20
30
40
50
60
70
80
90
100

From the values of L and N in the foregoing tables the logarithmic (or exponential) curve on the preceding page is constructed. As only positive values of N are considered, the curve never reaches the L-axis. The approximate values of L for numbers from .1 to 10 are measured from ON to the curve.

From the curve, $\log 2 = .3$; that is, $2 = 10^{.3}$.

Then $20 = 10 \cdot 2 = 10^{1} \cdot 10^{.3} = 10^{1.3}$.

Thus the logarithm of 20 is 1 greater than the logarithm of 2. Similarly, the logarithm of 30 is 1 greater than the logarithm of 3, and so on.

Therefore, if line $O'N'$ be drawn one unit below ON, the logarithms of numbers from 10 to 100 are the values of distances to the curve from points on $O'N'$ which correspond to these numbers. This practically gives us a considerable portion of the curve beyond point A.

EXERCISES

Find from the curve the logarithm of:

1. 2.
2. 3.
3. 4.
4. 5.
5. 6.2.
6. 7.4.
7. 9.6.
8. 11.
9. 15.
10. .2.
11. .5.
12. .9.
13. 25.
14. 50.
15. 64.
16. 96.
17. 100.
18. 10.
19. 1.
20. 32.

Find from the curve the number whose logarithm is:

21. .3.
22. .4.
23. .6.
24. .7.
25. .95.
26. $-.1$.
27. $-.4$.
28. $-.5$.
29. $-.25$.
30. 1.3.
31. 1.7.
32. 1.6.
33. 1.8.
34. 2.
35. .84.
36. 1.2.

The preceding exercises should familiarize the student with the meaning of the curve. We shall now use it to explain logarithmic multiplication, division, raising to a power, and extracting a root. It must be remembered that the curve is on too small a scale to give very close approximations. To draw a logarithmic curve which would give results sufficiently accurate for most practical purposes would require a piece of cross-ruled paper about two feet square.

180. Logarithmic multiplication. Multiplication by means of the logarithmic curve is illustrated in the

Example: Multiply 3 by 2.

Solution: From the curve, $\log 3 = .47$; hence $3 = 10^{.47}$.
From the curve, $\log 2 = .3$; hence $2 = 10^{.3}$.
Then $3 \cdot 2 = 10^{.47} \cdot 10^{.3} = 10^{.77}$.
From the curve, $10^{.77} = 6$.

EXERCISES

Compute as in the preceding example:

1. $2 \cdot 5$.
2. $3 \cdot 3$.
3. $4 \cdot 4$.
4. $.8 \cdot 8$.
5. $30 \cdot 2$.
6. $25 \cdot 4$.
7. $.5 \cdot 80$.
8. $22 \cdot 4.8$.
9. $14 \cdot 6$.
10. $.6 \cdot 80$.
11. $.7 \cdot 97$.
12. $1.5 \cdot 7.2$.

181. Logarithmic division. Division by means of the logarithmic curve is illustrated in the

Example: Divide 40 by 8.

Solution: From the curve, $\log 40 = 1.6$; hence $40 = 10^{1.6}$.
From the curve, $\log 8 = .9$; hence $8 = 10^{.9}$.
Then $40 \div 8 = 10^{1.6} \div 10^{.9} = 10^{.7}$.
From the curve, $10^{.7} = 5$.

EXERCISES

Compute as in the preceding example:

1. $8 \div 2$.
2. $6 \div 3$.
3. $40 \div 5$.
4. $18 \div 8$.
5. $16 \div .8$.
6. $48 \div 6$.
7. $22 \div 11$.
8. $56 \div 8$.
9. $\dfrac{4 \cdot 6}{3}$.
10. $\dfrac{80 \cdot 40}{25}$.

182. Logarithmic involution. A number is raised to a power by means of the logarithmic curve, as in the

Example: Find 2^3.

Solution: From the curve, $\log 2 = .3$; hence $2 = 10^{.3}$.

Therefore $2^3 = (10^{.3})^3 = 10^{.9}$.

From the curve, $10^{.9} = 8$.

EXERCISES

Compute as in the preceding example:

1. 3^2.
2. 5^2.
3. 3^3.
4. 4^3.
5. $\dfrac{5^2 \cdot 10^3}{25^2}$.
6. $\dfrac{2^2 \cdot 3^3}{6}$.
7. $4^2 \cdot 3^3$.

183. Logarithmic evolution. Roots are extracted by means of the logarithmic curve, as in the

Example: Find $\sqrt[3]{40}$.

Solution: From the curve, $\log 40 = 1.6$; hence $40 = 10^{1.6}$.

Therefore $\sqrt[3]{40} = (40)^{\frac{1}{3}} = (10^{1.6})^{\frac{1}{3}} = 10^{.53}$.

From the curve, $10^{.53} = 3.45$, which is approximately $\sqrt[3]{40}$.

EXERCISES

Compute as in the preceding example:

1. $\sqrt{3}$.
2. $\sqrt[3]{4}$.
3. $\sqrt[4]{81}$.
4. $6^{\frac{1}{2}}$.
5. $7^{\frac{1}{3}}$.
6. $5^2 \cdot \sqrt{3}$.
7. $\dfrac{3^3 \cdot \sqrt{2}}{.7}$.

From the foregoing work the student should see that *a logarithm is an exponent*, and that by the use of logarithms multiplication is effected by addition, division by subtraction, involution by a single multiplication, and evolution by a single division. The values of N and L, which up to this time have been taken from the curve, will hereafter be obtained much more accurately from the table on pages 448–449.

184. Steps preceding computation. Before computation by means of the table can be taken up, two processes requiring considerable explanation and practice must be mastered.

I. *To find from the table the logarithm of a given number.*

II. *To find from the table the number corresponding to a given logarithm.*

185. Base. If $N = b^L$, the logarithm of N to the **base** b is L. This last is expressed by the equation $\log_b N = L$. Therefore $N = b^L$ and $\log_b N = L$ are two ways of expressing the same fact.

Consequently $2 = 10^{.301}$ and $\log_{10} 2 = .301$ are equivalent statements.

The base of the **common** or **Briggs** system of logarithms is 10. The base 10 is often omitted. Thus log 2 means $\log_{10} 2$. This system is used in numerical work to the exclusion of all others.

The base of the *natural* system of logarithms is the irrational number 2.7182 +, which is usually denoted by e.

The natural system of logarithms is used for theoretical purposes only.

EXERCISES

Write in the notation of logarithms:

1. $300 = 10^{2.47}$.
2. $65 = 10^{1.81}$.
3. $4 = 10^{.60}$.
4. $1 = 10^{0}$.
5. $.10 = 10^{-1}$.
6. $1730 = 10^{3.238}$.
7. $173 = 10^{2.238}$.
8. $1.73 = 10^{.238}$.
9. $.173 = 10^{-1+.238}$.
10. $.0173 = 10^{-2+.238}$.

Write as powers of 10:

11. $\log 3 = .47$.
12. $\log 20 = 1.301$.
13. $\log 4.9 = .69$.
14. $\log 490 = 2.69$.
15. $\log .0049 = -3 + .69$.
16. $\log 381 = 2.58$.

186. Characteristic and mantissa. Unless a number is an exact power of 10, its logarithm consists of an *integer* and a *decimal*.

The integral part of a logarithm is called its **characteristic.**

The decimal part of a logarithm is called its **mantissa.**

The word "mantissa" means an addition; that is, a decimal portion which is added on to the integral part, or characteristic, of the logarithm. This term was used at one time to indicate the decimal part of any number, but for over a century it has been applied almost exclusively to logarithms.

Log 200 = 2.301. Here 2 is the characteristic and .301 is the mantissa.

Biographical Note. John Napier. Although many scientists have been honored with titles on account of their discoveries, very few of the titled aristocracy have become distinguished for their mathematical achievements. A notable exception to this rule is found in John Napier, Lord of Merchiston (1550–1617), who devoted most of his life to the problem of simplifying arithmetical operations.

Napier was a man of wide intellectual interests and great activity. In connection with the management of his estate he applied himself most seriously to the study of agriculture, and experimented with various kinds of fertilizer in a somewhat scientific manner, in order to find the most effective means of reclaiming soil. He spent several years in theological writing. When the danger of an invasion by Philip of Spain was imminent he invented several devices of war. Among these were powerful burning mirrors, and a sort of round musket-proof chariot, the motion of which was controlled by those within, and from which guns could be discharged through little portholes.

But by far the most serious activity of Napier's life was the effort to shorten the more tedious arithmetical processes. He invented the first approximation to a computing machine, and also devised a set of rods, often called Napier's bones, which were of assistance in multiplication. His crowning achievement, however, was the invention of logarithms, to which he devoted fully twenty years of his life.

No characteristics are given in the table on pages 248–249. The characteristic of any number is obtained from an inspection of the number itself according to rules which will now be derived.

$10^4 = 10000$; that is, the $\log 10000 = 4$.
$10^3 = 1000$; that is, the $\log 1000 = 3$.
$10^2 = 100$; that is, the $\log 100 = 2$.
$10^1 = 10$; that is, the $\log 10 = 1$.
$10^{-1} = .1$; that is, the $\log .1 = -1$.
$10^{-2} = .01$; that is, the $\log .01 = -2$.
$10^{-3} = .001$; that is, the $\log .001 = -3$.

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The preceding table indicates between what two integers the logarithm of a given number lies. This determines the characteristic.

Since 542 lies between 10^2 and 10^3, log $542 = 2$ plus a decimal.

And since .0045 lies between 10^{-3} and 10^{-2}, log $.0045 = -3$ plus a positive decimal (or -2 plus a negative decimal).

For the determination of the characteristic of a positive number we have the rules

I. *The characteristic of a number greater than 1 is one less than the number of digits to the left of the decimal point.*

II. *The characteristic of a number less than 1 is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.*

Accordingly the characteristic of 25 is 1; of 2536 is 3; of 6 is 0; of .4 is -1; of .032 is -2; of .00036 is -4.

The table on pages 248–249 gives the mantissas of numbers from 10 to 999. Before each mantissa a decimal point is understood.

The numbers 5420, 542, 5.42, .0542, and .000542 are spoken of as composed of the same *significant* digits in the same order. They differ only in the position of the decimal point, and consequently their logarithms will differ only in their characteristics. If the base of the system is 10, however, such numbers will have the same mantissa.

The last two points are easily illustrated by any two numbers which have the same significant digits in the same order.

$$\text{Log } 5.42 = .734, \text{ or } 5.42 = 10^{.734}.$$
$$5.42 \cdot 10^2 = 542 = 10^{.734} \cdot 10^2 = 10^{2.734}.$$

Therefore log $542 = 2 \cdot 734$.

The property just explained does not belong to a system of logarithms in which the base is any number other than 10. It is a very convenient property, as tables of a given accuracy are far shorter when the base is 10 than they would be with any other base. For example, the table on pages 248–249 gives the mantissas of all numbers from 1 to 999. But these mantissas are just the same as the mantissas of the three-figure decimals from .001 to .999, or another set of a thousand numbers. Were the base any number other than 10, the mantissas of the numbers from 1 to 999 would be different from those of the numbers from .001 to .999. Four pages or more would then be required to print a table equivalent to the one which is here put on two.

187. Use of the tables. To obtain the logarithm of a number of three or fewer significant figures from the tables, we have the

Rule. *Determine the characteristic by inspection.*

Find in column N *the first two significant figures. In the row with these and in the column headed by the third figure of the given number find the required mantissa.*

EXERCISES

Find the logarithm of:

1. 271.
2. 344.
3. 982.
4. 65.
5. 650.
6. 27.
7. 2.7.
8. 2700.
9. 3.
10. 6.
11. 932.
12. .932.

Solution: The characteristic of .932 is -1 and the mantissa is .9694. Hence $\log .932 = -1 + .9694$. This is usually written in the abbreviated form $\bar{1}.9694$. The mantissa is always kept positive in order to avoid the addition and subtraction of both positive and negative decimals, which in ordinary practice contain from three to five figures. Negative characteristics, being integers, are comparatively easy to take care of. (The student should note that log .932 is really negative, being $-1 + .9694$, or $-.0306$.)

13. .643.
14. .0532.
15. .00267.
16. .00579.
17. .0101.
18. 825000.

188. Interpolation. The process of finding the logarithm of a number not found in the table, from the logarithms of two numbers which are found there, or the reverse of this process, is called **interpolation.**

If we desire the logarithm of a number not in the table, say 7635, we know that it lies between the logarithms of 7630 and 7640, which are given in the table. Since 7635 is halfway between 7630 and 7640, we assume, though it is not *strictly* true, that the required logarithm is halfway between their logarithms, 3.8825 and 3.8831. To find log .7635 we first look up log 7630 and log 7640 and then take half (or .5) their difference (this difference may be taken from the column headed D) and add it to log 7630. This gives

$$\log 7635 = 3.8825 + .5 \times .0006 = 3.8828.$$

Were we finding log 7638, we should take .8 of the difference between log 7630 and log 7640 and add it to log 7630.

The preceding solution illustrates the general

RULE. *Prefix the proper characteristic to the mantissa of the first three significant figures.*

Then multiply the difference between this mantissa and the next greater mantissa in the table (called the tabular difference, column D *of the table) by the remaining figures of the number preceded by a decimal point.*

Add the product to the logarithm of the first three figures and reject all decimals beyond the fourth place.

In this method of interpolation we have assumed that the increase in the logarithm is directly proportional to the increase in the number. As has been said, this is not strictly true, yet the results here obtained are nearly always correct to the fourth decimal place.

EXERCISES

Find the logarithm of:

1. 4625.
2. 364.7.
3. 42.73.
4. 32.75.
5. 546.8.
6. 72.543.
7. 10.101.
8. 700.35.
9. 505.50.
10. 2.0075.
11. .00386.
12. .0007777.
13. 3.1416.
14. 2.71828.
15. .023456.

189. Antilogarithms. An **antilogarithm** is the number corresponding to a given logarithm. Thus antilog 2 equals 100.

If we desire the antilogarithm of a given logarithm, say 4.7308, we proceed as follows: The mantissa .7308 is found in the *row* which has 53 in column *N*, and in the *column* which has 8 at the top. Hence the first three significant figures of the antilogarithm are 538. Since the characteristic is 4, the number must have five digits to the left of the decimal point.

Thus antilog 4.7308 = 53,800. Therefore if the mantissa of a given logarithm is found in the table, its antilogarithm is obtained by the

RULE. *Find the row and the column in which the given mantissa lies.*

In the row found, take the two figures which are in column N for the first two significant figures of the antilogarithm, and for the third figure the number at the top of the column in which the mantissa stands.

Place the decimal point as indicated by the characteristic.

EXERCISES

Find the antilogarithm of:

1. 3.9309.

2. 1.8162.

3. .6284.

4. $\bar{1}.3541$.

5. $\bar{2}.5740$.

6. $8.5740 - 10$.

HINT. $8.5740 - 10 = \bar{2}.5740$.

7. $9.7292 - 10$.

8. $4.8136 - 10$.

9. 0.4533.

10. $\bar{4}.6345$.

11. 6.9232.

12. 8.2148.

13. $5.7832 - 6$.

14. $\bar{5}.9996$.

If the mantissa of a given logarithm, as 2.5271, is not in the table, the antilogarithm is obtained by interpolation as follows:

The mantissa 5271 lies just between

.5263, the mantissa of 336,

and .5276, the mantissa of 337.

Therefore the antilogarithm of 1.5271 lies between 33.6 and 33.7. Since the tabular difference is 13 and the difference between .5263 and .5271 is 8, the mantissa .5271 lies $\frac{8}{13}$ of the way from .5263 to .5276. Therefore the required antilogarithm lies $\frac{8}{13}$ of the way from 33.6 to 33.7.

Then antilog $1.5271 = 33.6 + \frac{8}{13} \times .1.$

$$33.6 + .061 = 33.66.$$

Therefore when the mantissa is not found in the table, we have the

Rule. *Write the number of three figures corresponding to the lesser of the two mantissas between which the given mantissa lies.*

Subtract the lesser mantissa from the given mantissa and divide the remainder by the tabular difference to one decimal place.

Annex this figure to the three already found and place the decimal point where indicated by the characteristic.

EXERCISES

Find the antilogarithms of:

1. 1.5723.
2. 2.3921.
3. 0.6690.
4. 2.5728.
5. $\bar{1}.2586$.
6. 7.3472 − 10.
7. 9.8527 − 10.
8. 5.9616 − 8.
9. 9.2654 − 10.
10. .7829.
11. 7.1050 − 10.
12. 6.2308 − 10.

190. Multiplication and division. Multiplication by logarithms depends on the

Theorem. *The logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

Proof. Let $\log_b N_1 = l_1$, (1)
and $\log_b N_2 = l_2$. (2)
From (1), $N_1 = b^{l_1}$. (3)
From (2), $N_2 = b^{l_2}$. (4)
(3) × (4), $N_1 N_2 = b^{l_1 + l_2}$. (5)
Therefore $\log_b N_1 N_2 = l_1 + l_2$.

Division by logarithms depends on the

THEOREM. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

$$\begin{array}{lcr}
\textit{Proof.}\text{ Let} & \log_b N_1 = l_1, & (1)\\
\text{and} & \log_b N_2 = l_2. & (2)\\
\text{From (1),} & N_1 = b^{l_1}. & (3)\\
\text{From (2),} & N_2 = b^{l_2}. & (4)\\
(3) \div (4), & \dfrac{N_1}{N_2} = b^{l_1 - l_2}. & \\
\text{Therefore} & \log_b \dfrac{N_1}{N_2} = l_1 - l_2. &
\end{array}$$

EXERCISES

Perform the indicated operation by logarithms:

1. $18 \cdot 25$.

$$\begin{array}{lr}
\textbf{Solution:} & \log 18 = 1.2553\\
 & \log 25 = 1.3979\\
\hline
\text{Adding,} & \log 18 \cdot 25 = 2.6532\\
\text{Antilog} & 2.6532 = 450.
\end{array}$$

2. $37 \cdot 23$.

3. $28 \cdot 8$.

4. $9.8 \cdot 5$.

5. $42 \cdot 2.2$.

6. $386 \cdot 27$.

7. $432 \cdot 361$.

8. $589 \cdot 734$.

9. $4326 \cdot 638$.

10. $2870 \cdot 3654$.

11. $286.7 \cdot 2.341$.

12. $3.412 \cdot 2.596$.

13. $432 \cdot .574$.

$$\begin{array}{lrcl}
\textbf{Solution:} & \log 432 = 2.6355 & = & 2.6355\\
 & \log .574 = \bar{1}.7589 & = & 9.7589 - 10\\
\hline
\text{Adding,} & \log 432 \cdot .574 = 2.3944 & = & 12.3944 - 10\\
\text{Antilog} & 2.3944 = 247.9. & &
\end{array}$$

Since the *mantissa is always positive,* any number carried over from the tenth's column to the units column is positive. This occurs in the preceding solution where $.6 + .7 = 1.3$, giving $+1$ to be added to the sum of the characteristics $+2$ and -1, in the units column. Mistakes in such cases will be few if the logarithms with negative characteristics be written as in the $9 - 10$ notation on the right.

In the preceding example and in others which follow, two methods are given for writing the logarithms which have negative characteristics. This is done to illustrate those cases in which the second of the two ways is preferable. It should be understood that in practice one, but not necessarily both, of these methods is to be used.

14. $385 \cdot .617$.
15. $541 \cdot .073$.
16. $37.6 \cdot .00835$.
17. $.0876 \cdot .673$.
18. $.07325 \cdot 6.384$.
19. $.6381 \cdot .01897$.
20. $675 \cdot .0236$.
21. $.437 \cdot .0076$.
22. $891 \div 27$.

Solution:

$$\begin{array}{lr} & \log 891 = 2.9499 \\ & \log 27 = 1.4314 \\ \hline \text{Subtracting,} & \log(891 \div 27) = 1.5185 \\ \text{Antilog} & 1.5185 = 33. \end{array}$$

23. $96 \div 12$.
24. $888 \div 37$.
25. $976 \div 321$.
26. $489 \div 27.1$.
27. $3460 \div 4.32$.
28. $4697 \div 281$.
29. $9876 \div 56.78$.
30. $6432 \div 7.81$.
31. $3.26 \div .0482$.

Solution:

$$\begin{array}{lrcr} & \log 3.26 = & 0.5132 = & 10.5132 - 10 \\ & \log .0482 = & \bar{2}.6830 = & 8.6830 - 10 \\ \hline \text{Subtracting,} & \log(3.26 \div .0482) = & 1.8302 = & 1.8302 - 0 \\ \text{Antilog} & & 1.8302 = 67.64. & \end{array}$$

32. $2.35 \div .0673$.
33. $4.86 \div .721$.
34. $.0635 \div .287$.
35. $.2674 \div 3.86$.
36. $7635 \div 8692$.
37. $.07382 \div 68.72$.
38. $\dfrac{256 \cdot 372}{128}$.
39. $\dfrac{347 \cdot (-625)}{346}$.
40. $\dfrac{463.2 \cdot 4.78}{-68.3}$.
41. $\dfrac{9.63 \cdot .0872}{.00635}$.
42. $.078 \div 4.267$.

191. Involution and evolution. Involution by logarithms depends on the

Theorem. *The logarithm of the mth power of a number is m times the logarithm of the number.*

Proof. Let $\log_b N = l$. (1)

Then $N = b^l$. (2)

Raising both members of (2) to the mth power,

$$N^m = b^{ml}.$$

Therefore $\log_b N^m = ml.$

Evolution by means of logarithms depends on the

THEOREM. *The logarithm of the real mth root of a number is the logarithm of the number divided by m.*

Proof. Let $\log_b N = l.$ (1)

Then $N = b^l.$ (2)

Extracting the mth root of the members of (2),

$$(N)^{\frac{1}{m}} = (b^l)^{\frac{1}{m}} = b^{\frac{l}{m}}. \quad (3)$$

Therefore $\log (N)^{\frac{1}{m}} = \frac{l}{m}\cdot$ (4)

EXERCISES

Compute, using logarithms:

1. $(2.73)^3$.

Solution: $\log 2.73 = .4362.$
Multiplying by 3, $\log (2.73)^3 = 1.3086.$
Antilog $1.3086 = 20.35.$

2. $(6.32)^4$. **3.** $(34.26)^2$. **4.** $(6.715)^3$. **5.** $(.425)^3$.

Solution: $\log .425 = \bar{1}.6284 = (9.6284 - 10).$
Multiplying by 3; $\log (.425)^3 = \bar{2}.8852 = (28.8852 - 30).$
Antilog $\bar{2}.8852 = .07676.$

Since the mantissa is always positive, we have in the preceding solution $+1$ (from $3 \cdot .6$) to unite with -3 (from $3 \cdot \bar{1}$). No confusion of positive and negative parts need arise, if the logarithms are written as indicated in the parenthesis.

6. $(.352)^4$. **7.** $(.0672)^2$. **8.** $(.003567)^5$. **9.** $\sqrt[3]{376}$.

Solution: $\log 376 = 2.5752.$
Dividing by 3, $\log \sqrt[3]{376} = .8584.$
Antilog $.8584 = 7.218 = \sqrt[3]{376}.$

10. $\sqrt[3]{583}$. **11.** $\sqrt[5]{1235}$. **12.** $\sqrt[3]{.000639}$.

Solution: $\log .000639 = \bar{4}.8055.$

If one divided $\bar{4}.8055$ as it stands by 3, he would be almost certain to confuse the negative characteristic and the positive mantissa. This and other difficulties may easily be avoided by adding to the characteristic and subtracting from the resulting logarithm any integral multiple of the index of the root which will make the characteristic positive.

Thus $\log .000639 = 2.8055 - 6.$

Dividing by 3, $\log \sqrt[3]{.000639} = .9351 - 2.$

Antilog $\bar{2}.9351 = .08612 = \sqrt[3]{.000639}.$

13. $\sqrt{.0786}.$ **15.** $\sqrt[4]{.002679}.$ **17.** $(4.965)^{\frac{3}{2}}.$

14. $\sqrt[5]{.0007324}.$ **16.** $(38.4)^{\frac{2}{3}}.$ **18.** $(-6.387)^{\frac{5}{3}}.$

19. $\sqrt{\dfrac{283 \cdot 4.627}{(8.423)^3}}.$

20. $\sqrt{\dfrac{(23.56)^2 \cdot 7.384}{(4.623)^3}}.$

21. $\sqrt[11]{209}.$

22. $\sqrt{87 - \sqrt[4]{163}}.$

23. $\frac{25}{361}\sqrt[5]{\frac{127}{67}\sqrt[3]{872}}.$

Note. The following four-place table will usually give results correct to one half of one per cent. Five-place tables give the mantissa to five decimal places of the numbers from 1 to 9999, and, by interpolation, the mantissa of numbers from 1 to 99999. Five-place tables give results correct to one twentieth of one per cent, an accuracy which is sufficient for most engineering work.

Six-place tables give the mantissa to six decimals for the same range of numbers as a five-place table. The labor of using a six-place table is about fifty per cent more than that of using a five-place one. For this reason and for other reasons a six-place table is of small practical value.

Seven-place tables give the mantissas of the numbers from 1 to 99999, and by interpolation give the mantissa of numbers from 1 to 999999. Seven-place tables are seldom needed in engineering, but are of constant use in astronomy.

In place of a table of logarithms engineers often use an instrument called a "slide rule." This is really a mechanical table of logarithms arranged ingeniously for rapid practical use. Results can be obtained with such an instrument far more quickly than with an ordinary table of logarithms, and that without recording or even thinking of a single logarithm. A "slide rule" ten inches long gives results correct to three figures. In work requiring greater accuracy a larger and more elaborate instrument which gives a five-figure accuracy is used.

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

PROBLEMS IN MENSURATION

Solve, using logarithms (obtain results to four figures):

1. The circumference of a circle is $2\pi R$. ($\pi = 3.1416$, R = radius.)

(a) Find the circumference of a circle whose radius is 42 inches.

(b) Find the radius of a circle whose circumference is 6843 centimeters.

2. The area of a circle is πR^2.

(a) Find the area of a circle whose radius is 3.672 feet.

(b) Find the radius of a circle whose area is 64.37 feet.

3. The area of the surface of a sphere is $4\pi R^2$.

(a) The radius of the earth is 3958.79 miles. Find its surface.

(b) Find the length of the equator.

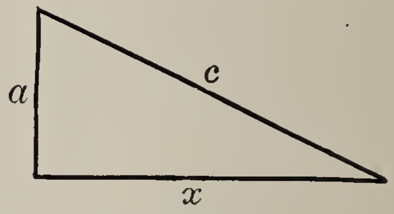
4. The volume of a sphere is $\dfrac{4\pi R^3}{3}$.

(a) Find the radius of a sphere whose volume is 25 cubic feet.

(b) Find the diameter of a sphere whose volume is 85 cubic inches.

5. If the hypotenuse and one leg of a right triangle are given, the other leg can always be computed by logarithms.

In the adjacent figure let a and c be given and x required.



Then $x = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)}$.

Whence $\log x = \frac{1}{2}\log(c+a) + \frac{1}{2}\log(c-a)$.

(a) The hypotenuse of a right triangle is 377 and one leg is 288. Find the other leg.

(b) The hypotenuse of a right triangle is 1493 and one leg is 532. Find the other leg.

6. The area of an equilateral triangle whose side is s is $\dfrac{s}{4}\sqrt{3}$.

(a) Find in square feet the area of an equilateral triangle whose side is 11.47 inches.

(*b*) Find the side of an equilateral triangle whose area is 60 square centimeters.

7. The area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$. Here a, b, and c are the sides of the triangle and $s = \dfrac{a+b+c}{2}$.

(*a*) Find the area of a triangle whose sides are 12 inches, 15 inches, and 19 inches respectively.

(*b*) Find the area of a triangle whose sides are 557, 840, and 1009.

192. Exponential equations. An **exponential** equation is an equation in which the unknown occurs as an exponent or in an exponent.

Many exponential equations are readily solved by means of logarithms, since $\log a^x = x \log a$. Thus let $a^x = c$. Then $x \log a = \log c$. Whence $x = \log c \div \log a$.

MISCELLANEOUS EXERCISES

Solve for x:

1. $8^x = 324$.

Solution: $\log 8^x = \log 324,$

or $x \cdot \log 8 = \log 324.$

Whence $x = \dfrac{\log 324}{\log 8} = \dfrac{2.5105}{.9031} = 2.75+.$

2. $3^x = 25$.

3. $64^x = 4$.

4. $16^x = 1024$.

5. $(-2)^x = 64$.

6. $3 = (1.04)^x$.

7. $2^x = 64$.

8. $4^{2x+1} = 84$.

9. $\dfrac{81^{\frac{1}{2}}}{3^{-x-1}} = 27^{\frac{2x-1}{3}}$.

10. $3^{x+7} = 5^x$.

11. In how many years will one dollar double itself at 3%, interest compounded annually?

Solution: At the end of one year the amount of \$1 at 3% is \$1.03; at the end of two years it is \$(1.03)(1.03) or $\$(1.03)^2$; at the end of three years it is $\$(1.03)^3$; and at the end of x years it is $\$(1.03)^x$.

If x is the number of years required, $(1.03)^x = 2$.

Taking the logarithms of both members of the equation,

$$x \log 1.03 = \log 2.$$

Solving, $$x = \frac{\log 2}{\log 1.03} = \frac{.3010}{.0128} = 23.5+.$$

12. In how many years will \$1 double itself at 5%, interest compounded annually?

13. In how many years will any sum of money treble itself at 4%, interest compounded annually?

14. In how many years will \$265 double itself at $3\frac{1}{2}$%, interest compounded annually?

15. In how many years will \$4000 amount to \$7360.80 at 5%, interest compounded annually?

16. About 300 years ago the Dutch paid \$24 for the island of Manhattan. At 4% compound interest, what would this payment amount to at the present time?

17. In how many years will \$12 double itself at 4%, interest compounded semiannually?

18. Show that the amount of P dollars in t years at r%, interest compounded annually, is $P(1+r)^t$; compounded semiannually is $P\left(1+\frac{r}{2}\right)^{2t}$; compounded quarterly is $P\left(1+\frac{r}{4}\right)^{?}$; and compounded monthly is $P\left(1+\frac{r}{?}\right)^{12t}$.

19. Find the amount of \$5000 at the end of four years, interest at 8% compounded (*a*) annually; (*b*) semiannually; (*c*) quarterly.

20. A man borrows \$6000 to build a house, agreeing to pay \$50 monthly until the principal and interest at 6% is paid. Find the number of full payments required.

21. If each payment in Exercise 20 is at once lent at 6%, compounded annually, what will they all amount to by the time the final payment of \$50 is made?

22. From Exercises 20 and 21 determine the total interest received by the money lender up to the time of the last payment. What rate per cent on the original $6000 is this?

Find the number of digits in:

23. (*a*) $2^9 \cdot 3^8 \cdot 5^7$; (*b*) 3^{52}; (*c*) 2^{340}.

24. Can the base of a system of logarithms be negative? Explain. What is the logarithm of -2? -10? $-n$?

Find (without reference to the table) the numerical values of:

25. $\log_3 9$.

26. $\log_2 8$.

27. $\log_8 2$.

28. $\log_9 27$.

29. $\log_{27} 9$.

30. $\log_4 8 + \log_8 4$.

31. $\log_{27} 81 - \log_{81} 27$.

32. $\log_{25} 125 + \log_5 25 - \log_{125} 5$.

33. $\log_3 (\frac{1}{3}) - \log_9 (\frac{1}{27}) + \log_{27} 9$.

Simplify:

34. $\log \frac{5}{8} + \log \frac{24}{25}$.

35. $\log \frac{7}{32} - \log \frac{35}{64}$.

36. $\log \frac{25}{4} + \log \frac{36}{50} - \log \frac{3}{4}$.

37. $2 \log 3 + 3 \log 2$.

38. $3 \log 4 + 4 \log 3 - 2 \log 6$.

Solve for x:

39. $a^x = c^{x-1}$.

40. $a^{x-1} = c^{x-2}$.

41. $a^{x-1} \cdot b^x = c^{2x}$.

42. $3^x \cdot 2^{\frac{1}{x}} = 6$.

43. $e^x = e^{-x}$.

44. $2^{2x} \cdot 3^{\frac{2}{x}} = 36$.

45. $a^{x+1} = b^{2x} \div c^{x-1}$.

46. $a^{4x} + 8a^{2x} = 6a^{3x}$.

47. $a^{5x} + a^{4x} = 6a^{4x} - 6a^{3x}$.

Solve for x and y:

48. $\begin{array}{l} 2^x = 3^y, \\ 3^{x-1} = 4^y. \end{array}$

49. $\begin{array}{l} 2x - y = 5, \\ 3^x \cdot 9^{3y} = 9^{11}. \end{array}$

50. $\begin{array}{l} 3x + y = 9, \\ 2^x \cdot 8^{2y} = 4^{10}. \end{array}$

51. $\begin{array}{l} 8^x \cdot 5^y = 50, \\ 2^{6x} \cdot 3^{2y} = 328. \end{array}$

52. $\begin{array}{l} 3^x - 6^y = 0, \\ 3^{x+1} - 6^x = 0. \end{array}$

53. $\begin{array}{l} x^y - y^x = 0, \\ y - x^2 = 0. \end{array}$

Note. It is not a little remarkable that just at the time when Galileo and Kepler were turning their attention to the laborious computation of the orbits of planets, Napier should be devising a method which simplifies these processes. It was said a hundred years ago, before astronomical computations became so complex as they now are, that the invention of logarithms, by shortening the labors, doubled the effective life of the astronomer. To-day the remark is well inside the truth.

In the presentation of the subject in modern textbooks a logarithm is defined as an exponent. But it was not from this point of view that they were first considered by Napier. In fact it was not till long after his time that the theory of exponents was understood clearly enough to admit of such application. This relation was noticed by the mathematician Euler, about one hundred fifty years after logarithms were invented.

It was by a comparison of the terms of certain arithmetical and geometrical progressions that Napier derived his logarithms. They were not exactly like those used commonly to-day, for the base which Napier used was not 10. Soon after the publication (1614) of Napier's work, Henry Briggs, an English professor, was so much impressed with its importance that he journeyed to Scotland to confer with Napier about the discovery. It is probable that they both saw the necessity of constructing a table for the base 10, and to this enormous task Briggs applied himself. With the exception of one gap, which was filled in by another computer, Briggs's tables form the basis for all the common logarithms which have appeared from that day to this.

CHAPTER XXXVIII

RATIO, PROPORTION, AND VARIATION

193. Ratio and proportion. The student should review the definitions and theorems of pages 176–183.

EXERCISES

Simplify the following ratios by writing them as fractions and reducing the fractions to lowest terms:

1. $42 : 28$.

2. $24\,a^3 : 56\,a^2$.

3. $(x^2 - y^2) : (x - y)$.

4. $(x^3 + 8\,y^3) : (x + 2\,y)$.

5. $\left(1 - \frac{4}{a^2}\right) : \left(1 - \frac{2}{a}\right)$.

6. $\left(a - \frac{16}{a}\right) : \left(\frac{24}{a^4} + \frac{10}{a^3} + \frac{1}{a^2}\right)$.

7. (*a*) 4 weeks : 12 hours; (*b*) 48,000 inches : 2 miles.

8. 1 mile : 1 kilometer. (1 meter = 39.37 inches.)

9. Separate 150 into three parts in the ratio 4 : 6 : 2.

10. If a is a positive number, which is the greater ratio:

(*a*) $\dfrac{5 + 3\,a}{5 + 4\,a}$ or $\dfrac{5 + 4\,a}{5 + 5\,a}$? (*b*) $\dfrac{7 - 2\,a}{7 - 3\,a}$ or $\dfrac{7 - 3\,a}{7 - 4\,a}$?

(*c*) If a positive number is added to or subtracted from both terms of a proper fraction, what change is produced in the numerical value of the fraction?

If $a : b = c : d$, prove the following and state the corresponding theorem in words:

11. $a : c = b : d$.

12. $b : a = d : c$.

13. $a^n : b^n = c^n : d^n$.

14. $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$.

15. $(a + b) : b = (c + d) : d$.

16. $(a + b) : a = (c + d) : c$.

17. $(a - b) : b = (c - d) : d$.

18. $(a - b) : a = (c - d) : c$.

19. $(a + b) : (a - b) = (c + d) : (c - d)$.

The results in Exercises 15, 17, and 19 are said to be derived from $a : b = c : d$ by **addition, subtraction,** and **addition and subtraction** respectively.

20. If $a : b = c : d = e : f$, prove that $(a + c + e) : (b + d + f) = a : b$ and state the theorem in words.

21. Find a mean proportional between 1.44 and .0256.

22. Find a third proportional to 15 and 125.

23. Find a fourth proportional to $16\frac{1}{4}$, $8\frac{1}{3}$, and $62\frac{1}{2}$.

24. Write $5 : 15 = 8 : 24$ by addition, subtraction, addition and subtraction, alternation, and inversion.

Solve:

25. $8 : 12 = (3 - x) : 7$.

26. $4 : x = x : 169$.

27. $3 : 5 = x^{-1} : 2$.

28. $8 : x = 12 : (10 - x)$.

29. $3 : 5 = (x - 3) : (2x + 18)$.

30. $20 : x = x : (10 - x)$.

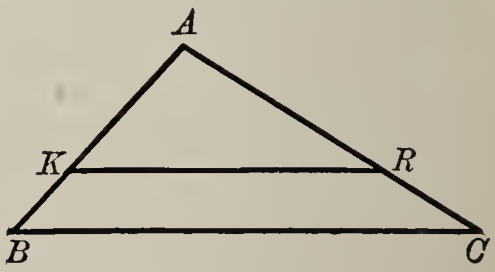
31. The surface, S, of a sphere is $4\pi R^2$. Prove that for any two spheres $\frac{S_1}{S_2} = \frac{R_1^2}{R_2^2} = \frac{D_1^2}{D_2^2}$, where D denotes the diameter.

32. The volume, V, of a sphere of radius R and diameter D is $\frac{4\pi R^3}{3}$. Prove that for any two spheres $\frac{V_1}{V_2} = \frac{R_1^3}{R_2^3} = \frac{D_1^3}{D_2^3}$.

33. Find the ratio of the surfaces of the earth and the moon, their diameters being 2160 miles and 7920 miles respectively.

34. Find the ratio of the volumes of the earth and the moon.

35. If ABC is any triangle, and KR is a line parallel to BC meeting AB at K and AC at R, then the area of ABC is to the area of AKR as $\overline{AB}^2 : \overline{AK}^2$, or as $\overline{AC}^2 : \overline{AR}^2$, or as $\overline{BC}^2 : \overline{KR}^2$.



If in the adjacent figure the area of ABC is 100 square inches, that of AKR is 25 square inches, and AB equals 10 inches, find AK.

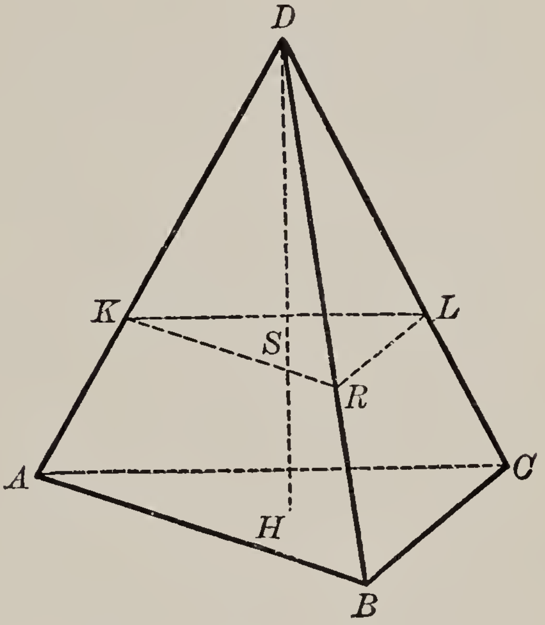
36. If in the figure of Exercise 35 $AB = 12$, and triangle AKR is $\frac{1}{2}$ triangle ABK, find AK.

37. If in Exercise 35 the trapezoid $KRCB$ is eight times as large as triangle AKR, and $AC = 40$, find AR.

38. If AB equals 32, and two parallels to BC separate triangle ABC into three parts of equal area, find to two decimals the lengths of the three parts into which AB is divided.

39. If a plane be passed parallel to the base of a pyramid (or cone) cutting it in KRL, then pyramid $D\text{-}ABC$: pyramid $D\text{-}KRL = \overline{DH}^3 : \overline{DS}^3$, etc.

If in the adjacent figure the volumes of the pyramids are 4 and 32 cubic inches respectively, and the altitude DH equals 18 inches, find DS.



40. If DH is 12 inches and the volume of one pyramid is one half the volume of the other, find DS to two decimals.

41. If the volume of the frustum is $\frac{19}{27}$ of the whole pyramid, and DH equals 36, find DS.

42. If two planes parallel to the base divide the whole pyramid into three parts having equal volumes, and $DH = 100$, find, using page 501, the parts into which the planes divide DH.

If $a : b = c : d$, prove:

43. $\dfrac{a + 3b}{a - 3b} = \dfrac{c + 3d}{c - 3d}$.

44. $\dfrac{a^2 + 2b^2}{a^2} = \dfrac{c^2 + 2d^2}{c^2}$.

45. $\dfrac{a^2 + b^2}{2ab} = \dfrac{c^2 + d^2}{2cd}$.

46. $\dfrac{5a^3 - b^3}{b^3} = \dfrac{5c^3 - d^3}{d^3}$.

47. $\dfrac{a^3 + b^3}{3a^2b + 3ab^2} = \dfrac{c^3 + d^3}{3c^2d + 3cd^2}$.

48. $\dfrac{a^2 - ab + b^2}{3ab} = \dfrac{c^2 - cd + d^2}{3cd}$.

49. In a right triangle h is the hypotenuse and a and b are the legs. The corresponding sides of another right triangle are H, A, and B. If $h : H = a : A$, prove $a : A = b : B$. Are the triangles similar?

Note. By the earlier mathematicians ratios were not treated as if they were numbers, and the equality of two ratios which we know as a proportion was not denoted by the same symbol as other kinds of equality. The usual sign of equality for ratios was : :, a notation which was introduced by the Englishman, Oughtred, in 1631, and brought into common use by John Wallis about 1686. The sign = was used in this connection by Leibnitz (1646–1716) in Germany, and by the continental writers generally, while the English clung to Oughtred's notation.

194. Variation. The word **quantity** denotes anything which is measurable, such as distance, rate, and area.

Many operations and problems in mathematics deal with numerical measures of quantities, some of which are fixed and others constantly changing.

An abstract number, or the numerical measure of a fixed quantity, is called a **constant**.

Thus the abstract numbers 1, 3, and $-\frac{5}{7}$ are constants. Any definite quantities, as the area of a square whose side is 2, the circumference of a circle divided by its diameter (3.1416 nearly), the time of one revolution of the earth on its axis (23^h, 56^m, 4.09^s), and the velocity of light through space (186,330 miles per second) are constants.

The numerical measure of a changing quantity is called a **variable**.

For example, the distance (measured in any unit of length) between a passenger on a moving car and a point on the track either ahead of or behind him is a variable, decreasing in the first instance, increasing in the second. Other examples of variables are one's weight, the height of the mercury in the thermometer, and the distance to the sun.

The equation $x = 3y$ may refer to no physical quantities whatever, yet it is possible to imagine y as taking on in succession every possible numerical value, and the value of x as accompanying every change, and consequently always being three times as great as the corresponding value of y. In this sense, which is strictly mathematical, x and y are variables.

Problems in variation deal with at least two variables so related that any change in one is accompanied by a change in the other. Frequently one variable depends on several others.

For instance, the number of lines of printing on a page depends on the distance between the lines, the size of the type, and one dimension of the page.

The symbol for variation is $\propto$, and $x \propto y$ is read x *varies directly as* y, or x *varies as* y.

195. Direct variation. One hundred feet of copper wire of a certain size weighs 32 pounds. Obviously a piece of the same kind 200 feet long would weigh 64 pounds; a piece 300 feet long would weigh 96 pounds, and so on.

Here we have two variables W (weight) and L (length) so related that the value of W depends on the value of L, and in such a way that W increases proportionately as L increases. That is, W is directly proportional, or merely proportional, to L. Hence, if W_1 and W_2 are *any* two weights corresponding to the lengths L_1 and L_2 respectively,

$$W_1 : W_2 = L_1 : L_2. \tag{1}$$

In the form of a variation (1) becomes

$$W \propto L.$$

In general, if $x \propto y$, and x and y denote *any* two corresponding values of the variables, and x_1 and y_1 a *particular* pair of corresponding values,

$$\frac{x}{x_1} = \frac{y}{y_1}. \tag{2}$$

From (2),

$$x = \left(\frac{x_1}{y_1}\right)y. \tag{3}$$

But $\frac{x_1}{y_1}$ is a constant, being the quotient of two definite numbers. Call this constant K and (3) may be written

$$x = Ky.$$

That is, if one variable varies as a second, the first equals the second multiplied by a constant.

Thus for the copper wire just mentioned, $W = \frac{32}{100} L$, or $\frac{8}{25} L$. Here, though W varies as L varies, W is always equal to L multiplied by the constant $\frac{8}{25}$.

The phrase **varies with** is often incorrectly used in place of **varies as.** The latter should be used to denote a *proportional* change in one variable with respect to a second; the former should not be so used. A boy's height varies *with* his age, but does not vary *as* his age. At 3 years the average boy is about 3 feet tall; at 12 years he is about 5 feet. At the latter time, if his height varied as his age from 3 years up to 12 years, he would be 12 feet tall.

196. Inverse variation. If a tank full of water is emptied in 24 minutes through a "smooth" outlet in which the area of the opening, A, is 1 square inch, an outlet in which A is 2 square inches would empty the tank twice as quickly, or in 12 minutes; and an outlet in which A is 3 square inches would empty the tank in 8 minutes.

Suppose it possible to increase or decrease A at will. We then have in t, the time required to empty the tank, and in A, the area of the opening, two related variables such that if A increases, t will decrease proportionally; while if A decreases, t will increase proportionally. That is, t and A are inversely proportional. This means that when A is doubled, t is halved; when A is trebled, t is divided by 3, and so on. The relation existing between the numerical values of A and t given in the preceding paragraph illustrates the truth of the last statement and of (1) which follows.

Now let t_1 and t_2 be *any* two times corresponding to the areas A_1 and A_2 respectively; then

$$t_1 : t_2 = A_2 : A_1. \qquad (1)$$

The letters and the subscripts in (1) say: *The first time is to the second time as the second area is to the first area.*

The proportion (1) may be put in another form.

First,
$$t_1 \cdot A_1 = t_2 \cdot A_2. \qquad (2)$$

Dividing (2) by A_1A_2, $$\frac{t_1}{A_2} = \frac{t_2}{A_1}, \tag{3}$$

or $$t_1\left(\frac{1}{A_2}\right) = t_2\left(\frac{1}{A_1}\right). \tag{4}$$

Whence $$t_1 : t_2 = \frac{1}{A_1} : \frac{1}{A_2}. \tag{5}$$

Here the subscripts on the t's and those on the A's come in the same order.

In the form of a variation (5) becomes $t \propto \frac{1}{A}$.

In general x varies **inversely** as y when x varies as the **reciprocal** of y; that is,

$$\boldsymbol{x \propto \frac{1}{y}}. \tag{6}$$

And if x and y denote any two corresponding values of the variable, and x_1 and y_1 a particular pair of corresponding values,

$$x : x_1 = \frac{1}{y} : \frac{1}{y_1}. \tag{7}$$

Whence $$\frac{x}{y_1} = \frac{x_1}{y}, \quad \text{or} \quad xy = x_1y_1. \tag{8}$$

But x_1y_1 is a constant, being the product of two definite numbers. Call this constant K.

Then (8) becomes $\boldsymbol{xy = K}$.

That is, if one variable varies inversely as another, the product of the two is a constant.

197. Joint variation. If the base of a triangle remains constant while the altitude varies, the area will vary as the altitude. Similarly, if the base varies while the altitude remains constant, the area will vary as the base. If *both* base and altitude vary, the area varies as the product of the two; that is, the area of the triangle varies **jointly** as the base and altitude. Further, if at *any* time A_1 denotes the area of a variable triangle, and h_1 and b_1 the corresponding altitude and base, and if A_2 denotes the area at *any other* time, and h_2 and b_2 the corresponding altitude and base, then $A_1 : A_2 = h_1b_1 : h_2b_2$.

In the form of a variation this last becomes

$$A \propto hb.$$

In general, any variable x varies jointly as two others, y and z, if

$$\boldsymbol{x \propto yz.} \tag{1}$$

If x varies jointly as y and z, and if x, y, and z denote *any* corresponding values of the variables, while x_1, y_1, and z_1 denote a *particular* set of such values, then

$$\boldsymbol{\frac{x}{x_1} = \frac{yz}{y_1 z_1}}. \tag{2}$$

From (2),

$$x = \left(\frac{x_1}{y_1 z_1}\right) yz. \tag{3}$$

But in (3) the fraction $\frac{x_1}{y_1 z_1}$ is a constant, since x_1, y_1, and z_1 are particular values of the variables x, y, and z. Calling this constant K, we may write $x \propto yz$ as the equation

$$\boldsymbol{x = Kyz.}$$

One variable may vary directly as one variable (or several variables) and inversely as another (or several others). Also one variable may vary as the square, or the cube, or the square root, or the reciprocal, or as any algebraic expression whatever involving the other variable (or variables).

The theory of variation is really involved in proportion, but this is not obvious to the beginner. Hence it is necessary to make clear the meaning of the terms used in variation, and to show how proportion is applied to the solution of problems in variation. It is doubly necessary that the student make this application, otherwise he will not readily grasp numerous relations in physics, in chemistry, and in astronomy; for many important laws of these sciences are often stated in the form of a variation. In connection with these laws many problems arise which require for their solution clear notions of the principles of variation. With a knowledge of proportion only, the student would find the laws vague and the problems difficult.

PROBLEMS

1. If $x \propto y$, and $x = 4$ when $y = 6$, find x when $y = 8$.

Solution: The variation is direct. Therefore

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}. \qquad (1)$$

Substituting in (1), $$\frac{4}{x_2} = \frac{6}{8}. \qquad (2)$$

Solving (2), $$x_2 = 5\tfrac{1}{3}.$$

2. If $x \propto y$, and $x = 6$ when $y = 10$, find y when $x = 15$.

3. If $x \propto y$, and $x = h$ when $y = k$, find y when $x = m$.

4. If x varies inversely as y, and $x = 6$ when $y = 7$, find x when $y = 21$.

Solution: The variation is inverse. Hence

$$x_1 : x_2 = \frac{1}{y_1} : \frac{1}{y_2}. \qquad (1)$$

Substituting in (1), $$6 : x_2 = \tfrac{1}{7} : \tfrac{1}{21}. \qquad (2)$$

Solving (2), $$x_2 = 2.$$

5. If $x \propto \frac{1}{y}$, and $x = 4$ when $y = 100$, find x when $y = 10$.

6. If $y \propto \frac{1}{z}$, and $y = h$ when $z = k$, find y when $z = m$.

7. If x varies jointly as y and z, and $x = 24$ when $y = 6$ and $z = 8$, find x when $y = 9$ and $z = 4$.

Solution: The variation is joint. Therefore

$$\frac{x_1}{x_2} = \frac{y_1 z_1}{y_2 z_2}. \qquad (1)$$

Substituting in (1), $$\frac{24}{x_2} = \frac{6 \cdot 8}{9 \cdot 4}, \text{ whence } x_2 = 18.$$

8. If x varies jointly as y and z, and $x = 3$ when $y = 4$ and $z = 5$, find x when $y = 20$ and $z = 2$.

9. If x varies directly as y and inversely as z, and $x = 10$ when $y = 4$ and $z = 9$, find x when $y = 2$ and $z = 6$.

HINT. Here $$x_1 : x_2 = \frac{y_1}{z_1} : \frac{y_2}{z_2}.$$

10. If d varies directly as t^2, and $d = 64$ when $t = 2$, find d when $t = 4$.

HINT. Here $$\frac{d_1}{d_2} = \frac{t_1^2}{t_2^2}.$$

11. If V varies directly as T and inversely as P, and $V = 80$ when $P = 15$ and $T = 400$, find P when $T = 450$ and $V = 45$.

12. The weight of any object below the surface of the earth varies directly as its distance from the center of the earth. An object weighs 100 pounds at the surface of the earth. What would be its weight (*a*) 1000 miles below the surface (radius of the earth = 4000 miles)? (*b*) 2000 miles below the surface? (*c*) at the center of the earth?

13. If a wagon wheel 4 feet 8 inches in diameter makes 360 revolutions in going a certain distance, how many revolutions will a wheel 5 feet in diameter make in going the same distance?

14. The distance which sound travels varies directly as the time. A man measures with a stop watch the time elapsing between the sight of the smoke from a hunter's gun and the sound of its report. When the hunter was 1 mile distant, the time was $4\frac{4}{5}$ seconds. How far off was the hunter when the observed time was 2 seconds?

15. When the volume of air in a bicycle pump is 24 cubic inches, the pressure on the handle is 30 pounds. Later, when the volume of air is 20 cubic inches, the pressure is 36 pounds. Assume that a proportion exists here, determine whether it is direct or inverse, and find the volume of the air when the pressure is 48 pounds.

16. The distance (in feet) through which a body falls from rest varies as the square of the time in seconds. If a body falls 16 feet in 1 second, how far will it fall in 6 seconds?

17. The intensity (brightness) of light varies inversely as the square of the distance from the source of the light. A reader holds his book 4 feet from a lamp, and later 6 feet distant. At which distance does the page appear brighter? How many times as bright?

18. A lamp shines on the page of a book 9 feet distant. Where must the book be held so that the page will receive four times as much light? twice as much light?

19. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. An object weighs 100 pounds at the surface of the earth. What would it weigh (*a*) 100 miles above the surface? (*b*) 2000 miles above the surface? (*c*) 4000 miles?

20. The area of any circle varies as the square of its radius. The area of a circle is 154 square inches. Its radius is 7 inches. Find the radius of a circle whose area is 594 square inches.

21. The weight of a sphere of given material varies directly as the cube of its radius. Two spheres of the same material have radii 2 inches and 6 inches respectively. The first weighs 6 pounds. Find the weight of the second.

22. The time required by a pendulum to make one vibration varies directly as the square root of its length. If a pendulum 100 centimeters long vibrates once in 1 second, find the time of one vibration of a pendulum 64 centimeters long.

23. Find the length of a pendulum which vibrates once in 2 seconds; once in 5 seconds.

24. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on 1 square foot is .9 pound when the rate of the wind is 15 miles per hour. Find the velocity of the wind when the pressure on 1 square yard is 18 pounds.

25. The pressure of water on the bottom of a containing vessel varies jointly with the area of the bottom and the depth of the water. When the water is 1 foot deep, the pressure on 1 square foot of the bottom is 62.5 pounds. (*a*) Find the pressure on the bottom of a tank 12 feet long and 8 feet wide in which the water is 6 feet deep. (*b*) Find the total pressure on one end and on one side.

26. The cost of ties for a railroad varies directly as the length of the road and inversely as the distance between the ties. The cost of ties for a certain piece of road, the ties being 2 feet apart, was $1320. Find the cost of ties for a piece twenty times as long as the first, if the ties are $2\frac{1}{2}$ feet apart.

27. It has been shown that if one variable varies as another, the second multiplied by a constant number equals the first. It is often desirable to determine this constant. Suppose such to be the case in Problem 16.

Solution: Since $d \propto t^2$, $d = Kt^2$ (K being some constant). (2)
But when $t = 2$ and $d = 64$, substituting in (2) gives

$$64 = K(2)^2 = 4K; \text{ whence } K = 16.$$

28. In Problem 21 $W \propto r^3$; hence $W = Kr^3$. Find K.

29. In Problem 22 $t \propto \sqrt{l}$. Find the constant which multiplied by $\sqrt{l}$ gives t.

30. In Problem 14 find the constant connecting d and t in the equation $d = Kt$ and determine its practical meaning.

31. It has been shown that if one variable varies inversely as a second, the product of the two is a constant. Find this constant in Problem 15.

32. The area of a triangle varies jointly as its base and altitude. What is the constant involved?

33. The area of a circle varies as the square of the diameter. What is the constant involved?

34. The volume of a sphere varies as the cube of the diameter. What is the constant involved?

35. Give concrete illustrations of direct, inverse, and joint variation different from those given in this book.

36. If $x^2 + y^2 \propto x^2 - y^2$, prove $x + y \propto x - y$.

37. If $x^3 + y^3 \propto x^3 - y^3$, prove $x + y \propto x - y$.

38. If $x + y \propto x - y$, prove $x^2 - xy + y^2 \propto x^2 + xy + y^2$.

CHAPTER XXXIX

IMAGINARIES

198. Definitions. When the square root of a negative number arose in our previous work, it was called an **imaginary,** and no attempt was then made to explain its meaning. The treatment of imaginaries was deferred because there were so many topics of more importance to the beginner. It must not be supposed, however, that imaginaries are not of great value in mathematics. They are also of much use in certain branches of applied science; and it is unfortunate that symbols which can be used in numerical computations to obtain practical results should ever have been called imaginary.

The equation $x^2+1=0$, or $x^2=-1$, asks the question, "What is the number whose square is -1?" By defining a new number, $\sqrt{-1}$, as a number whose square is -1, we obtain one root for the equation $x^2+1=0$. Similarly, $\sqrt{-5}$ is a number whose square is -5. And, in general, $\sqrt{-n}$ is a number whose square is $-n$. Obviously $\sqrt{-5}$ means something very different from $\sqrt{5}$.

The positive numbers are all multiples of the unit $+1$, and the negative numbers are all multiples of the unit -1. Similarly, **pure imaginary** numbers are real multiples of the imaginary unit $\sqrt{-1}$.

Thus $\sqrt{-1}+\sqrt{-1}=2\sqrt{-1}$, and $\sqrt{-1}+2\sqrt{-1}=3\sqrt{-1}$, etc.
Further, $\sqrt{-4}=2\sqrt{-1}$; $\sqrt{-a^2}=a\sqrt{-1}$; $\sqrt{-5}=\sqrt{5}\sqrt{-1}$.

The imaginary unit $\sqrt{-1}$ is often denoted by the letter i; that is, $3\sqrt{-1}=3i$.

If a real number be united to a pure imaginary by a plus sign or a minus sign, the expression is called a **complex number.**

7. $\sqrt{-18}+\sqrt{-8}.$

8. $(-12)^{\frac{1}{2}}+(-27)^{\frac{1}{2}}.$

9. $3+2\sqrt{-1}+5-6\sqrt{-1}.$

10. $5\sqrt{-x^2}-7a-3\sqrt{-x^2}.$

11. $6-2\sqrt{-64x^2}-3\sqrt{-25x^2}+8.$

12. $5\sqrt{-3}+3\sqrt{-2}-\sqrt{-27}+2\sqrt{-8}.$

13. $6\sqrt{-4a^4}-7a^2\sqrt{-9}+3\sqrt{-6}-5\sqrt{-24}.$

14. $(12-6\sqrt{-9})-(15+2\sqrt{-36}).$

15. $3a-2x-(2a\sqrt{-a^2}-5ra^2\sqrt{-1}).$

16. $(x-iy)-(n-iv).$

Write as a multiple of $\sqrt{-1}$:

17. $\sqrt{-10}.$

18. $\sqrt{-6}.$

19. $2\sqrt{-3}.$

20. $\sqrt{-a}.$

21. $a\sqrt{-b}.$

22. $\sqrt{-a-b}.$

200. Multiplication of imaginaries. By the definition of square root, the square of $\sqrt{-n}$ is $-n$.

Therefore

$$(\sqrt{-1})^2=-1.$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=-1\sqrt{-1}.$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=1.$$

To multiply $\sqrt{-2}$ by $\sqrt{-3}$ we write $\sqrt{-2}$ as $\sqrt{2}\cdot\sqrt{-1}$, and $\sqrt{-3}$ as $\sqrt{3}\cdot\sqrt{-1}$.

Then
$$\begin{aligned}\sqrt{-2}\cdot\sqrt{-3}&=(\sqrt{2}\cdot\sqrt{-1})(\sqrt{3}\cdot\sqrt{-1})\\&=\sqrt{6}\cdot\sqrt{-1}\cdot\sqrt{-1}=-\sqrt{6}.\end{aligned}$$

In general, if $\sqrt{-a}$ and $\sqrt{-b}$ are two imaginaries whose product (or quotient) is desired, they should first be written in the form $\sqrt{a}\cdot\sqrt{-1}$ and $\sqrt{b}\cdot\sqrt{-1}$, and the multiplication (or division) should then be performed. It must be clearly understood that the rule on page 247 for the multiplication of radicals does not apply to imaginaries.

Thus $\sqrt{2}\cdot\sqrt{3}=\sqrt{2\cdot 3}=\sqrt{6}.$

But $\sqrt{-2}\cdot\sqrt{-3}$ **does not equal** $\sqrt{(-2)(-3)}$, which equals $\sqrt{6}.$

In multiplying two complex numbers, write each expression in the form $a + bi$ and proceed as in the following

Example. Multiply $2 + \sqrt{-3}$ by $3 - \sqrt{-7}$.

Solution:

$$\begin{array}{ll} & 2 + \sqrt{-3} = 2 + \sqrt{3} \cdot \sqrt{-1} \\ & 3 - \sqrt{-7} = 3 - \sqrt{7} \cdot \sqrt{-1} \\ \hline \text{Multiplying,} & 6 + 3\sqrt{3}\sqrt{-1} - 2\sqrt{7}\sqrt{-1} + \sqrt{21} \\ \text{Rewriting,} & 6 + 3\sqrt{-3} - 2\sqrt{-7} + \sqrt{21} \end{array}$$

EXERCISES

Perform the indicated multiplications and simplify results:

1. $(\sqrt{-1})^5$.

2. $(\sqrt{-1})^6$.

3. $(\sqrt{-1})^7$.

4. $(\sqrt{-1})^8$.

5. $2\sqrt{-1} \cdot 3\sqrt{-1}$.

6. $\sqrt{-9} \cdot \sqrt{-16}$.

7. $\sqrt{-5}\,(-\sqrt{-6})$.

8. $\sqrt{-25} \cdot \sqrt{3}$.

9. $2\sqrt{-3} \cdot 3\sqrt{-2}$.

10. $\sqrt{-m} \cdot \sqrt{-n}$.

11. $4\sqrt{-5}\,(-3\sqrt{-6})$.

12. $\sqrt{a+b} \cdot \sqrt{-a-b}$.

13. $(2 + \sqrt{-1})(2 - \sqrt{-1})$.

14. $(3 + \sqrt{-2})(3 - \sqrt{-2})$.

15. $(4 - 2\sqrt{3}\,i)(4 + 2\sqrt{3}\,i)$.

16. $(3 + \sqrt{-1})(6 - \sqrt{-2})$.

17. $(4 - 2i)(3 - 2\sqrt{3}\,i)$.

18. $(a + ib)(c + id)$.

19. $(a + ib)(a + ib)$.

20. $(a + bi)(a - bi)$.

21. $(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})^2$.

22. $(-\frac{1}{2} - \frac{1}{2}\sqrt{-3})^2$.

23. $(x - iy)^2 - (x + iy)^2$.

24. $(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3 - (-\frac{1}{2} - \frac{1}{2}\sqrt{-3})^3$.

25. $(a + i\sqrt{1 - x^2})(a - i\sqrt{1 - x^2})$.

26. What kind of a number is the indicated even root of any negative number? Explain.

Note. Long before the time of Gauss mathematicians had performed the operations of multiplication and division on complex numbers by the same rules that they used for real numbers. As early as 1545 Cardan stated that the product of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ was 40. However, he was not always equally fortunate in obtaining correct results, for in another place he sets $\frac{1}{4}\left(-\sqrt{-\frac{1}{4}}\right) = \frac{1}{\sqrt{64}} = \frac{1}{8}$.

Even the rather complicated formula for extracting any root of a complex number was discovered in the early part of the eighteenth century. But all of these operations were purely formal, and seemed to most mathematicians a mere juggling with symbols until Gauss showed clearly the place and usefulness of such numbers.

201. Division of imaginaries. One complex number is the conjugate of another if their product is *real.* Thus $a + bi$ and $a - bi$ are **conjugates.** Conjugate complex numbers are used in division of imaginary expressions as conjugate radicals are used in division of radicals.

Division by an imaginary is performed by writing the dividend over the divisor as a fraction and then multiplying both numerator and denominator by the simplest imaginary expression which will make the resulting denominator real and rational.

EXAMPLES

1. $\sqrt{-6} \div \sqrt{-2}$.

Solution:

$$\sqrt{-6} \div \sqrt{-2} = \frac{\sqrt{-6}}{\sqrt{-2}} = \frac{\sqrt{-6} \cdot \sqrt{-2}}{\sqrt{-2} \cdot \sqrt{-2}}$$

$$= \frac{\sqrt{6} \cdot \sqrt{-1} \cdot \sqrt{2} \cdot \sqrt{-1}}{\sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{2}\sqrt{-1}} = \frac{-\sqrt{12}}{-2} = \sqrt{3}.$$

2. $3 \div (2 + \sqrt{-3})$.

Solution:

$$3 \div (2 + \sqrt{-3}) = \frac{3}{2 + \sqrt{-3}} = \frac{3(2 - \sqrt{-3})}{(2 + \sqrt{-3})(2 - \sqrt{-3})}$$

$$= \frac{6 - 3\sqrt{-3}}{4 + 3} = \frac{6 - 3\sqrt{-3}}{7}.$$

EXERCISES

Perform the indicated division:

1. $\sqrt{-8} \div \sqrt{-2}$.
2. $\sqrt{-6} \div \sqrt{-3}$.
3. $2\sqrt{-3} \div 3\sqrt{-1}$.
4. $\sqrt[4]{-1} \div \sqrt[4]{-4}$.
5. $\sqrt{6} \div \sqrt{-2}$.
6. $(-25)^{\frac{1}{4}} \div (-81)^{\frac{1}{4}}$.
7. $\sqrt{ax} \div \sqrt{-x}$.
8. $\sqrt{-a} \div \sqrt{-b}$.
9. $(-5ax)^{\frac{1}{2}} \div (-2x)^{\frac{1}{2}}$.
10. $[(-a^6)^{\frac{1}{4}} - (-a^2)^{\frac{1}{4}}] \div (-a)^{\frac{1}{4}}$.

11. $2 \div (1 - \sqrt{-1})$.

12. $3 \div (2 - \sqrt{-2})$.

13. $2\sqrt{-1} \div (\sqrt{-1} + 3)$.

14. $3\sqrt{-2} \div (2\sqrt{-3} + 2)$.

15. $\dfrac{-1 + \sqrt{-3}}{-1 - \sqrt{-3}}$.

16. $\dfrac{1 + i}{1 - i}$.

17. $\dfrac{a}{a + bi}$.

18. $\dfrac{a + ib}{c + id}$.

19. $(2 + 3i) \div (2i - 1)(5i - 3) = ?$

20. Is $i\sqrt{3} - 1$ a cube root of 8?

21. Is $1 - \sqrt{-3}$ a cube root of -8?

22. Does $x^2 - 4x + 7 = 0$, if $x = 2 \pm \sqrt{-3}$?

23. Does $x = \frac{8}{5}\sqrt{-10}$, $y = -\frac{3}{5}\sqrt{-10}$ satisfy the system $x^2 - xy - 12y^2 = 8$, $x^2 + xy - 10y^2 = 20$?

24. Determine whether the sum and the product of $2 + 3\sqrt{-1}$ and $2 - 3\sqrt{-1}$ are real numbers.

25. Show that the *sum* and the *product* of any two conjugate complex numbers is *real*.

26. Show that the *quotient* of two conjugate complex numbers is *complex*.

27. Point out the error in the following:

The equation $\sqrt{x - y} = i\sqrt{y - x}$ is an identity. (1)

Let $x = a$ and $y = b$, and (1) becomes

$$\sqrt{a - b} = i\sqrt{b - a}. \qquad (2)$$

Now let $x = b$ and $y = a$, and (1) becomes

$$\sqrt{b - a} = i\sqrt{a - b}. \qquad (3)$$

From (2) and (3),

$$\sqrt{a - b} \cdot \sqrt{b - a} = i^2(\sqrt{b - a} \cdot \sqrt{a - b}). \qquad (4)$$

Whence $1 = i^2$, or $1 = -1$.

202. Equations with imaginary roots. The student should now be able to solve and check equations which have imaginary roots.

EXERCISES

Solve and check the equations which follow:

1. $x^2 + 4x + 12 = 0$.

2. $x^2 - 6x + 36 = 0$.

3. $x^2 + 5x + 7 = 0$.

4. $x^2 - 3x + 10 = 0$.

5. $2x^2 + 6x + 5 = 0$.

6. $3x^2 - 7x + 6 = 0$.

7. $x^3 = 1$.

HINT. If $x^3 = 1$, $x^3 - 1 = 0$.
Hence $(x - 1)(x^2 + x + 1) = 0$.
Then $x - 1 = 0$,
and $x^2 + x + 1 = 0$, etc.

8. $x^3 = 8$.

9. $x^3 = 27$.

10. $x^3 = -8$.

11. $x^4 = 1$.

12. $x^4 = 16$.

13. $x^6 = 1$.

14. $x^6 = 64$.

15. How many square roots has any real number? cube roots? fourth roots? sixth roots?

16. What do Exercises 7–14 indicate regarding the number of *n*th roots which any real number may have?

17. $8x^3 - 27 = 0$.

18. $125x^3 + 64 = 0$.

19. $(x^2 + 5)(x^2 - 7) + 27 = 0$.

20. $x^3 - x^2 + 2x - 2 = 0$.

21. $x^6 + 7x^3 - 8 = 0$.

22. $4x^4 + 20x^2 + 21 = 0$.

23. $64x^4 - 12x^2 - 27 = 0$.

24. $9x^4 + 18x^2 + 8 = 0$.

25. $50x^4 + 135x^2 + 36 = 0$.

26. $(x^2 + 9)(x^2 + 2x + 8) = 0$.

27. $(x^2 + x)^2 + 13(x^2 + x) + 36 = 0$.

28. $(x^2 + 5x)^2 + 17(x^2 + 5x) + 66 = 0$.

29. Solve $x + y = 4$, $x^2 - 3xy - y^2 = -39$ and check.

30. Solve $z^2 + x^2 = 130$, $z + x + 2\sqrt{z + x} = 2$ and check.

31. Solve Exercise 4, page 394, and check.

32. Solve Exercise 25, page 402, and check.

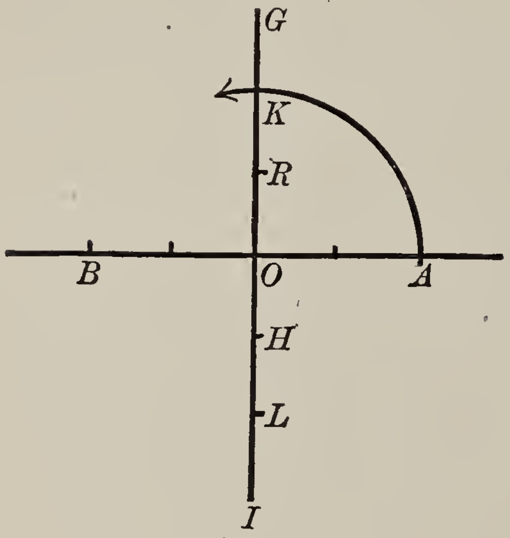
203. Factors involving imaginaries. After studying radicals we enlarged our previous notion of a factor, and, with certain limitations, employed radicals among the terms of a factor. Now in a similar manner, with like restrictions, we extend our notion of a factor still farther and use imaginary numbers as coefficients or as terms in a factor.

Thus $x^2+1=x^2-(-1)=(x+\sqrt{-1})(x-\sqrt{-1})$.
Similarly, $4x^2+9=4x^2-(-9)=(2x+3\sqrt{-1})(2x-3\sqrt{-1})$,
and $x^2+6=x^2-(-6)=(x+\sqrt{-6})(x-\sqrt{-6})$.

Further, $x^3-1=(x-1)(x^2+x+1)$. Hitherto the trinomial x^2+x+1 has been regarded as prime; but the student can easily prove that $x^2+x+1=(x+\frac{1}{2}+\frac{1}{2}\sqrt{-3})(x+\frac{1}{2}-\frac{1}{2}\sqrt{-3})$. Therefore x^3-1 has three factors, $x-1$, $x+\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $x+\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

If the student is curious as to the way in which the factors of x^2+x+1 were found, he may discover the method for himself by studying the results of Exercise 7, page 473.

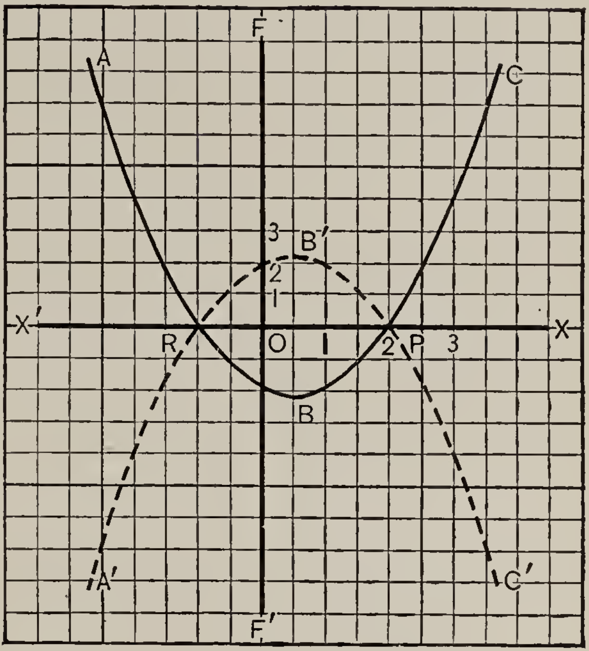
204. Graphical interpretation of pure imaginaries. In our previous graphical work a positive number and a numerically equal negative number, as $+2$ and -2, were represented by equal distances measured in opposite directions, such as *OA* and *OB* of the adjacent figure. Now multiplying $+2$ by -1 gives -2. Hence, if we choose to do so, we may regard -1 as an operator (rotor) which turns *OA* in the direction of the arrow into the position *OB*, or through two right angles (180 degrees).



To make this point clearer note the two curves of the figure on page 475. Curve *ABC* is the graph of the function x^2-x-2. Curve *A'B'C'* is the graph of the function $2+x-x^2$. The latter function was obtained by multiplying x^2-x-2 by -1. The graphical effect of this multiplication is to turn the whole curve *ABC* about *X'X* as an axis through *two right angles* to the position *A'B'C'*.

The preceding illustrations indicate a method of interpreting the $\sqrt{-1}$ which is in strict conformity with our previous graphical work.

First $\sqrt{-1}\cdot\sqrt{-1}=-1$. Now multiplying a number by -1 produces the same effect as multiplying twice in succession by $\sqrt{-1}$. Therefore multiplying by $\sqrt{-1}$ once may be regarded as producing a rotation of one right angle (90 degrees), or one half as much rotation as multiplying by -1.

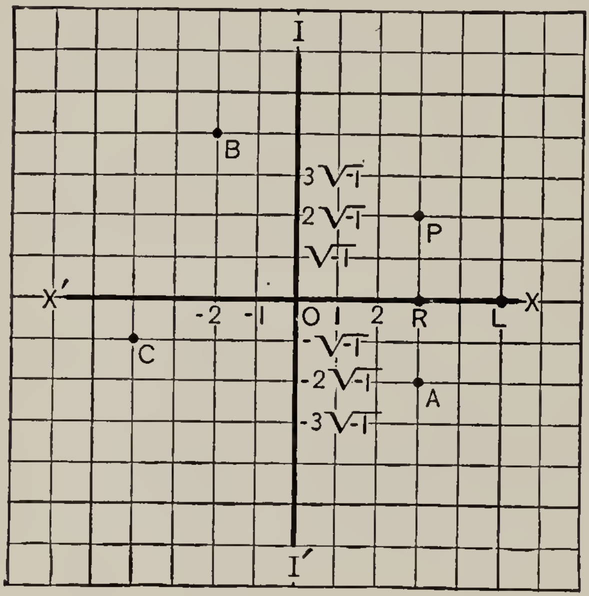


Returning to the figure on page 474, OA, or $+2$, multiplied by $\sqrt{-1}$ would be turned to the position OK. Hence the point K is said to correspond to the number $2\sqrt{-1}$. Similarly, point R corresponds to $\sqrt{-1}$ and point G to $3\sqrt{-1}$. And OH being measured in a direction opposite to OR, OK, and OG, would correspond to $-\sqrt{-1}$. In like manner OL corresponds to $-2\sqrt{-1}$. This last result, however, may be reached differently. Multiplying 2 by $\sqrt{-1}$ three times gives $-2\sqrt{-1}$. These successive multiplications by $\sqrt{-1}$ may be regarded as producing a counterclockwise rotation through three right angles, which would locate the point corresponding to $-2\sqrt{-1}$ on OI at L as before.

Therefore the graphical representation of a *pure imaginary* number $b\sqrt{-1}$ is by a point on an axis perpendicular (at right angles) to the axis of real numbers, b units in the direction of OG if b is positive, b units in the direction of OI if b is negative. This new axis will be called the imaginary or I-axis.

205. Graphical representation of a complex number. The complex number $x=3+2\sqrt{-1}$ consists of a real part 3 and the imaginary part $2\sqrt{-1}$. To represent such a number we measure in the following figure 3 units along OK from O to R, and then 2 units parallel to the imaginary axis II' from R. This gives the point P, which is the graphical representation of the complex number

$$x = 3 + 2\sqrt{-1}.$$



If the student pays proper attention to signs, he should now see that the point A corresponds to $x = 3 - 2\sqrt{-1}$, B to $x = -2 + 4\sqrt{-1}$, and C to $x = -4 - \sqrt{-1}$.

In general, if x is a complex number $a + bi$, x is represented by a point a units from the imaginary axis and b units from the real axis, the positive and negative directions being as indicated in the adjacent figure.

Note. It was the discovery of a graphical interpretation for the imaginary numbers which did more than anything else to make them mean something to students of mathematics. Until this discovery they were tolerated because their appearance as the roots of equations was a constant reminder of their existence. But they were usually regarded as meaningless, and the less one had to do with them the better he liked it. About 1800 a Norwegian by the name of Wessel, and the Frenchman, Argand, gave practically the same graphical interpretation as that found in the text, but their work was little noticed till Gauss adopted the method and, by his influence and ability, placed the imaginary number on a firm basis.

EXERCISES

Locate the point x if:

1. $x = 2 + i.$

2. $x = 3 + 3i.$

3. $x = 4 - 2i.$

4. $x = 1 - 3i.$

5. $x = -2 + i.$

6. $x = -3 - 2i.$

7. $x = 4 - i.$

8. $x = 4 + 4i.$

9. $x = -1 - i.$

10. $x = -3 - 5i.$

11. $x = 2 \pm \sqrt{-3}.$

12. $x = 3 \pm 3\sqrt{-2}.$

13. $x = 2 + \sqrt{-11}.$

14. $x = 2 - \sqrt{-11}.$

15. $x = \sqrt{-12} - 5.$

Note on use of imaginaries. We have explained the laws of addition, subtraction, multiplication, and division for imaginary (and complex) numbers and have made some use of them. It is largely because imaginaries obey these laws that we call them numbers, for it must be admitted that we cannot count objects with imaginary numbers. Nor can we state by means of them our age, our weight, or the area of the earth's surface. It should be remembered, however, that we can do none of these things with negative numbers. We may have a group of objects — books, for example — whose number is 5; but no group of *objects* exists whose number is -5, or -3, or any negative number whatever. If it be asked, How, then, can negative numbers and imaginary numbers have any practical use? the answer is this: They have a practical use because when they enter into our calculations and we have performed the necessary operations upon them and obtained our final result, that result can frequently be interpreted as a concrete number such as is dealt with in ordinary arithmetic. Moreover, if the result cannot be so interpreted, it is, in applied mathematics at least, finally rejected.

In that part of electrical engineering where the theory and measurement of alternating currents of electricity are treated, complex numbers have had extensive use. Their employment in the difficult problems which there arise has given a briefer, a more direct, and a more general treatment than the earlier ones where such numbers are not used.

In theoretical mathematics complex numbers have been of great value in many ways. For example, numerous important theorems about functions are more easily proved under the assumption that the variable is complex. Then, by letting the imaginary part of the complex number become zero, we obtain the proof of the theorem for real values of the variable. Indeed, the student need not go very far beyond this point in his mathematical work to learn that, if e is $2.7182+$ (see page 437), $e^{\sqrt{-1}}+e^{-\sqrt{-1}}$ is equal to the real number $1.028+$. At the same time he will learn also how such a form arises, and something of its importance. In a way which we cannot now explain, even so involved an expression as $(a+ib)^{c+id}$ has in higher work a meaning and a use. If the student pursues his mathematical studies far enough, that meaning and use, and a multitude of other uses for complex numbers, will become familiar to him. But the numbers which we have learned to use in this book, namely fractions, negative numbers, irrational numbers, and complex numbers, complete the number system of ordinary algebra, for, from the fundamental operations, no other forms of number can arise.

CHAPTER XL

THEORY OF QUADRATIC EQUATIONS

206. Character of the roots of a quadratic equation. It is often desirable to determine the character of the roots of a quadratic without actually solving it. To determine the character of the roots of an equation means to find out whether the roots are real or imaginary, rational or irrational, equal or unequal. These properties of the roots of a given quadratic depend on the three coefficients, which correspond to a, b, and c, in the general quadratic equation $ax^2 + bx + c = 0$. The solution of this equation gives the roots:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

This expression $b^2 - 4ac$ which occurs in each root is called the **discriminant** of the quadratic. If a, b, and c are rational numbers, it is evident from an inspection of the discriminant where it occurs in the values of r_1 and r_2, that the following statements are true:

I. *If* $b^2 - 4ac$ *is positive and not a perfect square, the roots are real, unequal, and irrational.*

II. *If* $b^2 - 4ac$ *is positive and a perfect square, the roots are real, unequal, and rational.*

III. *If* $b^2 - 4ac$ *is zero, the roots are equal.*

In this case there is really but one root, $\frac{-b}{2a}$.

IV. *If* $b^2 - 4ac$ *is negative, the roots are imaginary.*

EXERCISES

Determine the character of the roots of the following equations by the use of the discriminant:

1. $2x^2 + 5x - 6 = 0$.

Solution: $b^2 - 4ac = (5)^2 - 4 \cdot 2 \cdot (-6) = 25 + 48 = 73$.
Therefore the roots are real, unequal, and irrational.

2. $x^2 - 5x + 6 = 0$.

3. $5x^2 - 11x + 2 = 0$.

4. $4x^2 - 20x + 25 = 0$.

5. $5x^2 - 3x - 3 = 0$.

6. $7x^2 - 2x + 10 = 0$.

7. $x^2 - 6x + 6 = 0$.

8. $4x^2 = 9 - 9x$.

9. $5x = x^2 + 5$.

10. $x^2 - 5x + 7 = 0$.

11. $12x^2 - 7x + 6 = 0$.

12. $x(x - 5) = x - 16$.

13. $\frac{3}{x^2} + \frac{11}{x} = 20$.

14. $5x - \frac{11}{7} - \frac{6}{7x} = 0$.

Determine the values of K which will make the roots of the following equations equal. (To say the roots of a quadratic are equal is the usual mathematical way of stating that *the equation has but one root.*)

15. $x^2 - Kx + 16 = 0$.

Solution: $a = 1,\ b = -K,\ c = 16$.
Hence $b^2 - 4ac = K^2 - 64$.
In order that the roots be equal, $b^2 - 4ac$ must equal zero.
Therefore $K^2 - 64 = 0$.
Whence $K = \pm 8$.

Check: Substituting 8 for K in the original equation,

$$x^2 - 8x + 16 = 0.$$

Whence $x = 4$, only.
Similarly, substituting $K = -8$, $x^2 + 8x + 16 = 0$.
Whence $x = -4$, only.

16. $x^2 - Kx + 36 = 0$.

17. $x^2 - 3Kx + 81 = 0$.

18. $2x^2 + 4Kx + 98 = 0$.

19. $x^2 - 10x + K = 0$.

20. $2x^2 + 8x + K = 0$.

21. $9x^2 + 30x + K + 9 = 0$.

22. $4Kx^2 - 60x + 25 = 0.$ **24.** $49x^2 - (K+3)x + 4 = 0.$

23. $9K^2x^2 - 84x + 49 = 0.$ **25.** $(K^2+5)x^2 - 30x + 25 = 0.$

26. $(K^2+17)x^2 + (5K-4)x + 4 = 0.$

Determine the relation between h and k which will make the roots of the following equations equal:

27. $k^2x^2 + 6hx + 9 = 0.$ **29.** $x^2 + 4kx + 4h = 0.$

28. $kx^2 - 2hx + 16 = 0.$ **30.** $kx^2 - 2hx + 6 = 0.$

Determine the values of a for which the following systems will have two sets of equal roots:

31. $\begin{aligned} y^2 &= ax, \\ y &= x+1. \end{aligned}$ **33.** $\begin{aligned} x^2 + y^2 &= a^2, \\ y &= x+1. \end{aligned}$

32. $\begin{aligned} y^2 &= 2x, \\ y &= x+a. \end{aligned}$ **34.** $\begin{aligned} x^2 + y^2 &= 2x, \\ y &= x+a. \end{aligned}$

207. Relations between the roots and the coefficients. The roots of $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad (1)$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \qquad (2)$$

(1) + (2) gives $r_1 + r_2 = \dfrac{-2b}{2a} = -\dfrac{b}{a}.$ (3)

(1) × (2) gives $r_1r_2 = \dfrac{b^2 - (b^2 - 4ac)}{4a^2} = \dfrac{c}{a}.$ (4)

The general quadratic equation may be written

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \qquad (5)$$

Then for any quadratic in which the coefficient of x^2 is 1:

I. *From* (3) *and* (5) *the coefficient of* x, $\frac{b}{a}$, *is the sum of the roots with the sign changed.*

II. *From* (4) *and* (5) *the constant term,* $\frac{c}{a}$, *is the product of the roots.*

I and II may be used to form a quadratic whose roots are given.

EXERCISES

Form the quadratic whose roots are:

1. $5, -3$.

Solution: $r_1 + r_2$ with the sign changed $= -2$.

$$r_1 \times r_2 = 5(-3) = -15.$$

Hence the required equation is $x^2 - 2x - 15 = 0$.

2. $2, 7$.
3. $-3, 10$.
4. $-4, -5$.
5. $-12, -1$.
6. $\frac{2}{3}, 5$.
7. $10, -\frac{1}{5}$.
8. $2+\sqrt{3}, 2-\sqrt{3}$.
9. $-3 \pm \sqrt{5}$.
10. $\frac{4}{3} \pm \sqrt{7}$.
11. $\frac{3}{2} \pm \frac{1}{2}\sqrt{6}$.
12. $\dfrac{-6 \pm 2\sqrt{3}}{5}$.
13. $\sqrt{5}, -3\sqrt{5}$.
14. $3-\sqrt{2}, 2+\sqrt{2}$.
15. a, c.
16. $a, \dfrac{1}{a}$.
17. $3a, \dfrac{2a}{3}$.
18. $a+1, \dfrac{1}{a-1}$.

Solve the following equations and check each by showing that the sum of the roots with its sign changed is the coefficient of x, and that the product of the roots is the constant term:

19. $x^2 - 12x - 13 = 0$.
20. $x^2 - 10x + 16 = 0$.
21. $x^2 + 3x + 3 = 0$.
22. $x^2 - 5x + 20 = 0$.
23. $x^2 + 2x + 2 = 0$.

24. One root of $x^2 - 4x - 12 = 0$ is -2. Find the other root.

Solution: Let r_2 be the required root.

Then $$-(r_1 + r_2) = -(-2 + r_2) = -4.$$

Solving, $$r_2 = 6.$$

Check: $$r_1 r_2 = (-2)(6) = -12.$$

25. One root of $x^2 + 7x - 18 = 0$ is -9. Find the other root.

Find the value of the literal coefficient in the following:

26. $x^2 + 2x - c = 0$, if one root is 3.
27. $x^2 - x - c = 0$, if one root is 10.
28. $x^2 + 8x - c = 0$, if one root is -2.
29. $x^2 - cx - 70 = 0$, if one root is 10.
30. $x^2 + 2bx + 25 = 0$, if one root is -5.

31. $x^2 - 3ax - 52 = 0$, if one root is 4.

32. $2x^2 - 11x + c = 0$, if one root is 5.

33. $ax^2 - 20x + 12 = 0$, if one root is $\frac{2}{3}$.

34. $ax^2 - 6x - 21 = 0$, if one root is -3.

35. $x^2 - 8x + c = 0$, if one root is three times the other.

36. $x^2 + 7x + c = 0$, if one root exceeds the other by 1.

37. $x^2 + 11x + b = 0$, if the difference between the roots is 9.

38. $x^2 - 5x - c = 0$, if the difference between the roots is 7.

39. $x^2 - 5x - a = 0$, if the difference between the roots is -13.

208. Number of roots of a quadratic. Up to this we have assumed that a quadratic equation has but two roots. This fact can be proved from the preceding work as follows:

If we write the equation $ax^2 + bx + c = 0$ in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ and substitute therein from (3) and (4) on page 480, we get $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. This can be factored and written as $(x - r_1)(x - r_2) = 0$. Now if any value of x different from r_1 and r_2, say r_3, be a root of this equation, such a value when substituted for x must satisfy the equation $(x - r_1)(x - r_2) = 0$.

Hence $(r_3 - r_1)(r_3 - r_2)$ must equal zero. By definition, however, r_3 is different from r_1 and r_2. Consequently neither the factor $(r_3 - r_1)$ nor $(r_3 - r_2)$ can equal zero, and therefore their product cannot equal zero. This proves that no additional value, r_3, can satisfy the equation $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. As this equation is but another form of $ax^2 + bx + c = 0$, the latter has only two roots.

209. Formation of equations with given roots. The method of forming quadratic equations which was used in the preceding exercise applies to equations having two roots only. A reversal of the method of solving equations by factoring (page 124), however, enables us to build up an equation with any number of given roots.

The correctness of the method for three given roots will be clear from what follows:

Form the equation whose roots are a, b, and c.

Write $(x - a)(x - b)(x - c) = 0.$ (1)

Now if a is put for x in (1), we obtain $0 = 0$.

Similarly, if b or c is put for x in (1), the result is $0 = 0$.

Therefore (1) is the equation whose roots are a, b, and c, and the expanded form of (1), $x^3 - (a + b + c)x^2 + (ab + ac + bc)x + abc = 0$, is the required equation.

The same reasoning applies if we form in this way an equation with any number of given roots.

Note. The relations between the roots and the coefficients of an equation were discovered at about the same time by Vieta in France, by Girard in Holland, and by Harriot in England. Vieta actually wrote the cubic equation in about the form given in the text so as to display these relations. The very important algebraical theories which result from these properties were developed in detail by Newton, and have been the subject of study by many of the most distinguished mathematicians since his time.

EXAMPLES

1. Form the equation whose roots are 3 and -5.

Solution: By the conditions, $x = 3$ and $x = -5$.

Therefore $x - 3 = 0$ and $x + 5 = 0$.

Then $(x - 3)(x + 5) = 0$. (1)

Expanding, $x^2 + 2x - 15 = 0$. (2)

Substitution shows that 3 and -5 are the roots of (1) and (2).

2. Form the equation whose roots are 1, 3, and -2.

Solution: As before, $x - 1 = 0$, $x - 3 = 0$, and $x + 2 = 0$.

Therefore $(x - 1)(x - 3)(x + 2) = 0$. (1)

Expanding, $x^3 - 2x^2 - 5x + 6 = 0$. (2)

Inspection shows that the given roots 1, 3, and -2 satisfy the equations (1) and (2).

EXERCISES

By the method used in the preceding examples form the equation whose roots are:

1. 3, 7.

2. 4, -5, 6.

3. $1 + \sqrt{3}$, $1 - \sqrt{3}$.

4. $2 \pm \sqrt{5}$.

5. $\dfrac{3 \pm \sqrt{7}}{2}$.

6. 3, -3, 8.

7. 1, $\frac{3}{2}$, -2.

8. $1 \pm \sqrt{3}$, 3.

9. $a + b,\ a - b$.

10. $\frac{1}{a},\ 5a$.

11. $3c \pm \sqrt{2a}$.

12. $\frac{4a \pm \sqrt{3c}}{2}$.

13. $r_1,\ r_2,\ r_3$.

14. $3,\ 2 \pm \sqrt{a}$.

15. $-5,\ -7,\ 6,\ 8$.

16. $2 \pm \sqrt{3},\ 3 \pm \sqrt{2}$.

17. $1,\ -2,\ a \pm \sqrt{a}$.

210. Factors of quadratic expressions. Let r_1 and r_2 be the roots of $ax^2 + bx + c = 0$.

$$\text{Then } x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (r_1 + r_2)x + r_1r_2 = (x - r_1)(x - r_2),$$

$$\text{or} \qquad ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Therefore the three factors a, $x - r_1$, and $x - r_2$ of any quadratic expression can be written if we first set the expression equal to zero and solve for r_1 and r_2 the equation thus formed. Obviously the character of the roots so obtained will determine the character of the factors. Hence by the use of the discriminant $b^2 - 4ac$ we can decide whether the factors of a quadratic *expression* are real or imaginary, rational or irrational, without factoring it.

EXERCISES

Determine which of the following expressions have rational factors:

1. $x^2 - 3x - 40$.
2. $2x^2 + 5x - 7$.
3. $7x^2 - 9x + 18$.
4. $24x^2 - x - 10$.
5. $72x^2 - 17x + 1$.
6. $5x^2 + 3x - 20$.
7. $3x^2 - 9x + 28$.
8. $33h^2 - 233h - 6$.
9. $x^2 - 2ax + (a^2 - b^2)$.
10. $abx^2 - (b^2 + a^2)x + ab$.

Separate into rational, irrational, or imaginary factors:

11. $2x^2 + 5x - 8$.

Solution: Let $2x^2 + 5x - 8 = 0$.

Solving by formula, $x = \frac{-5 \pm \sqrt{25 - (-64)}}{4} = \frac{-5 \pm \sqrt{89}}{4}$.

Then $\quad r_1 = \dfrac{-5+\sqrt{89}}{4}$ and $r_2 = \dfrac{-5-\sqrt{89}}{4}.$

Therefore $2x^2+5x-8 = 2\left[x - \dfrac{-5+\sqrt{89}}{4}\right]\left[x - \dfrac{-5-\sqrt{89}}{4}\right]$

$$= \tfrac{1}{8}(4x+5-\sqrt{89})(4x+5+\sqrt{89}).$$

12. $x^2 - 7x - 30.$

13. $x^2 - 4x - 1.$

14. $x^2 + 2x + 2.$

15. $x^2 + 4x - 9.$

16. $4x^2 - 12x - 9.$

17. $25x^2 + 20x + 4.$

18. $6x^2 + 14x - 40.$

19. $10 - 9x - 9x^2.$

20. $10x^2 + 12 - 26x.$

21. $x^2 + 7x + 8.$

22. $x^2 + x + 1.$

23. $x^2 + 1.$

24. $x^2 + 9.$

25. $x^2 - 2ax + a^2 - b.$

26. $x^2 + 6ax + 9a^2 - 4b.$

27. $4x^2 + 4ax + a^2 - 4c.$

28. $x^2 - 4ax + 4a^2 + c.$

29. $ax^2 + bx + c.$

30. $x^2 - xy + 5x - 2y + 6.$

Solution: Let $x^2 - xy + 5x - 2y + 6 = 0.$

Then $\quad x^2 + (5-y)x - 2y + 6 = 0.$

Solving for x in terms of y by the formula,

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 4(-2y+6)}}{2}$$

$$= \frac{-5+y \pm \sqrt{y^2-2y+1}}{2}.$$

Whence $\quad x = -2$ and $y - 3.$

Therefore $x^2 - xy + 5x - 2y + 6 = (x+2)(x-y+3).$

31. $3x^2 - 6xy + 14x - 4y + 8.$

32. $x^2 - xy - 2y^2 + 3x - 6y.$

33. $x^2 - 4xy - y + 3y^2 - 2 - x.$

34. $x^2 - 2y^2 - xy + 2x + 5y - 3.$

35. $6x^2 + xy - 12y^2 - x + 10y - 2.$

CHAPTER XLI

THE BINOMIAL THEOREM

211. Powers of binomials. The following identities are easily obtained by actual multiplication:

$$(a+b)^2 = a^2 + 2\,ab + b^2. \tag{1}$$

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3. \tag{2}$$

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4. \tag{3}$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5. \tag{4}$$

If $a + b$ is replaced by $a - b$, the even-numbered terms in each of the preceding expressions will then be negative and the odd-numbered terms will be positive.

212. The expansion of $(a + b)^n$. The form of the expansion for the general case will now be indicated:

The first term is a^n *and the last is* b^n.

The second term is $na^{n-1}b$.

The exponents of a *decrease by 1 in each term after the first.*

The exponents of b *increase by 1 in each term after the second.*

The product of the coefficient of any term and the exponent of a *in that term, divided by the exponent of* b *increased by 1, gives the coefficient of the next term.*

The sign of each term of the expansion is + if a *and* b *are positive; the sign of the odd-numbered terms is − if* b *only is negative.*

The number of terms in the expansion is $n + 1$.

According to the rule,

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n. \tag{1}$$

The preceding expansion expresses in symbols the law known as the **binomial theorem.** The theorem holds for all positive values of n and with certain limitations (see § 215) for negative values as well. This will be assumed without proof.

Note. The coefficients of the various terms in the binomial expansion are displayed in a most elegant form as follows:

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

In this arrangement each row is derived from the one above it by observing that each number is equal to the sum of the two numbers, one to the right and the other to the left of it, in the line above. Thus $4 = 1 + 3$, $6 = 3 + 3$, etc. The next line is 1 5 10 10 5 1. The successive lines of this table give the coefficients for the expansions of $(a + b)^n$ for the various values of n. Thus the numbers in the last line of the triangle are seen to be the coefficients when $n = 4$; the next line would give those for $n = 5$. This arrangement is known as Pascal's Triangle, and was published in 1665. It was probably known to Tartaglia nearly a hundred years before its discovery by Pascal.

213. The factorial notation. The notation $5!$, or $\underline{|5}$, signifies $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, or 120. Similarly, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

In general, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)n$.

The sign $n!$, or $\underline{|n}$, is read **factorial n.**

With the factorial notation the denominators of the third and fourth terms in the expansion of $(a + b)^n$ in § 212 become $2!$ and $3!$ respectively.

EXERCISES

Expand by the rule:

1. $(a + b)^6$.
2. $(a - 1)^6$.
3. $(a + 1)^7$.
4. $(a + 2)^6$.
5. $(a + 3)^7$.
6. $(2 - a)^6$.

Obtain the first four terms of:

7. $(a + b)^{20}$.
8. $(a + b)^{30}$.
9. $(a + 1)^{40}$.
10. $(a - 2)^{20}$.

Expand:

11. $(a^2+2b)^5$. HINT. To avoid confusion of exponents first write

$$(a^2)^5 + \quad (a^2)^4(2b)^1 + \quad (a^2)^3(2b)^2 + \quad (a^2)^2(2b)^3 + \quad (a^2)^1(2b)^4 + (2b)^5.$$

Then in the spaces left for them put in the coefficients according to the rule of § 212.

Finally, expand, and simplify each term.

12. $(a^2-2)^6$.

13. $(a^2+2b)^7$.

14. $\left(a^2+\frac{1}{b}\right)^5$.

15. $\left(a^2-\frac{1}{a^3}\right)^6$.

Obtain in simplest form the first four terms of:

16. $(a^2+2b)^{20}$.

17. $\left(a^2-\frac{2}{a}\right)^{30}$.

18. $\left(\frac{a}{b}+\frac{3b}{a}\right)^{20}$.

19. $\left(\frac{2x}{y^3}-\frac{y^4}{6x^5}\right)^7$.

20. $(a^2-3b^2)^{10}$.

21. $\left(\frac{3x^5}{y^3}-\frac{2y^{15}}{9x^{12}}\right)^6$.

22. $\left(\frac{a^2}{b^3}-\frac{2b^2}{a^4}\right)^{12}$.

23. $\left(\frac{\sqrt{x}}{y}+\frac{\sqrt{y}}{x}\right)^{12}$.

24. Write the first six terms of the expansion of $(a+b)^n$ and test it for $n=1$, $n=2$, $n=3$, and $n=4$. How does the number of terms compare with n? What is the value of each coefficient after the $(n+1)$st? Why does not the expansion extend to more than six terms when $n=5$?

25. Write the first four terms of $\left(1+\frac{1}{n}\right)^n$.

Compute the following, correct to two decimal places:

26. $(1.1)^{10}$. HINT. $(1.1)^{10}=(1+.1)^{10}$, etc.

27. $(.98)^{11}$. HINT. $(.98)^{11}=(1-.02)^{11}$, etc.

28. $(2.9)^8$.

29. $(1.06)^6$.

30. $6!$.

31. $2!\cdot 4!$.

32. $6!\div 3!$.

33. $4!-3!\cdot 2!\,2!$.

34. $2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}$.

35. $\frac{15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{7!}$.

214. Extraction of roots by the binomial theorem. By reference to the expansion (1), page 486, it can be seen that none of the factors n, $n-1$, $n-2$, $n-3$, etc. become zero for fractional or negative values of n. Hence for such exponents the development of $(a+b)^n$ becomes an infinite series. If a is numerically greater than b, and n has any one of the values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., the resulting series has a limiting value. In those expansions where a is considerably greater than b, this value can be readily approximated by finding the sum of the first few terms. Therefore the square root, cube root, and all other roots can be obtained approximately by the aid of the binomial theorem.

Note. The process of extracting the square root and even the cube root by means of the binomial expansion was familiar to the Hindus more than a thousand years ago. The German, Stifel (1486–1567), stated the binomial theorem for all powers up to the seventeenth, and also extracted roots of numbers by this method.

EXAMPLES

Find to three decimals by the binomial theorem:

1. $(27)^{\frac{1}{2}}$.

Solution:
$$\begin{aligned}(27)^{\frac{1}{2}} &= (25+2)^{\frac{1}{2}}\\ &= 25^{\frac{1}{2}} + \tfrac{1}{2}\cdot 25^{-\frac{1}{2}}\cdot 2 - \tfrac{1}{8}\cdot 25^{-\frac{3}{2}}\cdot 2^2 + \tfrac{1}{16}\cdot 25^{-\frac{5}{2}}\cdot 2^3 \cdots\\ &= 5 + \tfrac{1}{5} - \tfrac{1}{250} + \tfrac{1}{6250}\cdots\\ &= 5 + .2 - .004 + .00016 = 5.196+.\end{aligned}$$

2. $(67)^{\frac{1}{3}}$.

Solution:
$$\begin{aligned}(67)^{\frac{1}{3}} &= (64+3)^{\frac{1}{3}}\\ &= 64^{\frac{1}{3}} + \tfrac{1}{3}\cdot 64^{-\frac{2}{3}}\cdot 3 - \tfrac{1}{9}\cdot 64^{-\frac{5}{3}}\cdot 3^2 + \tfrac{5}{81}\cdot 64^{-\frac{8}{3}}\cdot 3^3 \cdots\\ &= 4 + \tfrac{1}{16} - \tfrac{1}{1024} +\\ &= 4 + .0625 - .00097 = 4.0615.\end{aligned}$$

Here three terms give the result correct to five figures.

3. $(79)^{\frac{1}{2}}$.

Hint. $(79)^{\frac{1}{2}} = (81-2)^{\frac{1}{2}} = 81^{\frac{1}{2}} - \frac{1}{2}\cdot 81^{-\frac{1}{2}}\cdot 2 + \frac{1}{8}\cdot 81^{-\frac{3}{2}}\cdot 2^2 + \cdots.$

Here $(81-2)^{\frac{1}{2}}$ yields more accurate results with fewer terms than does $(64+15)^{\frac{1}{2}}$.

EXERCISES

Find to two decimals by the binomial theorem:

1. $(26)^{\frac{1}{2}}$. **3.** $(79)^{\frac{1}{2}}$. **5.** $(28)^{\frac{1}{3}}$. **7.** $(25)^{\frac{1}{3}}$.
2. $(38)^{\frac{1}{2}}$. **4.** $(120)^{\frac{1}{2}}$. **6.** $(66)^{\frac{1}{3}}$. **8.** $(720)^{\frac{1}{3}}$.

Find the first four terms of:

9. $(1+x)^{\frac{1}{2}}$. **11.** $(3-x)^{\frac{1}{2}}$. **13.** $(2+x)^{\frac{1}{3}}$.
10. $(2+x)^{\frac{1}{2}}$. **12.** $(1+x)^{\frac{1}{3}}$. **14.** $(3-x)^{\frac{1}{3}}$.

215. Limitations on a and b in $(a+b)^n$. The expansion $(a+b)^n$ has a meaning for all values of n, only if a and b are properly chosen. To illustrate the truth of this statement we shall consider the expansion of $(1+x)^{-1}$ for various values of x. By the theorem,

$$(1+x)^{-1}=1-x+x^2-x^3+x^4-x^5+x^6-\cdots. \qquad (1)$$

Now $(1+x)^{-1}=\dfrac{1}{1+x}\cdot$ Hence the left member of (1) has a meaning for all values of x except -1. The right member of (1) is an infinite geometrical series whose ratio is $-x$. This series has a limiting value only when x is numerically less than 1 (see page 421); that is, if x be a positive or negative proper fraction. For positive or negative values of x numerically greater than 1 the series has no definite value. Therefore the expansion has a meaning only when x is numerically less than 1. Here 1 corresponds to a and x to b in $(a+b)^n$, and the preceding discussion *indicates* but does not prove the truth of the following statement:

The expansion $(a+b)^n$ has a definite value if n is positive or negative, integral or fractional, provided a is greater than b.

A proof of this last statement is beyond the scope of this book.

Note. The binomial theorem occupies a remarkable place in the history of mathematics. By means of it Napier was led to the discovery of logarithms, and its use was of the greatest assistance to

Newton in making his most wonderful mathematical discoveries. But to-day the results of Napier and of Newton are explained without even so much as a mention of the binomial theorem, for simpler methods of obtaining these results have been discovered.

It was Newton who first recognized the truth of the theorem, not only for the case where n is a positive integer, which had long been familiar, but for fractional and negative values as well. He did not give a demonstration of the general validity of the binomial development, and none even passably satisfactory was given until that of Euler (1707–1783). The first entirely satisfactory proof of this difficult theorem was given by the brilliant young Norwegian, Abel (1802–1829).

216. The rth term of $(a+b)^n$. According to the binomial theorem the fifth term of the expansion (1) on page 486 is

$$\frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{1\cdot 2\cdot 3\cdot 4}.$$

If we note particularly this term and those on page 486, we can write down, from the considerations which follow, any required term without writing other terms of the expansion.

The *denominator of the coefficient* of the fifth term is $4!$. From the law of formation the denominator in the sixth term would be $5!$, in the seventh term $6!$, etc. Consequently in the rth term the denominator of the coefficient would be $(r-1)!$.

The *numerator of the coefficient* of the fifth term contains the product of the four factors $n(n-1)(n-2)(n-3)$. The sixth term would contain these four and the factor $n-4$. Similarly, the last factor in the seventh term would be $n-5$, etc. Hence the last factor in the rth term would be $n-(r-2)$. Therefore the numerator of the coefficient of the rth term is $n(n-1)(n-2)(n-3)\cdots(n-r+2)$.

The *exponent of a* in the fifth term is $n-4$, in the sixth term it would be $n-5$, etc. Therefore in the rth term the exponent of a would be $n-(r-1)$ or $n-r+1$.

The *exponent of b* in the fifth term is 4, in the sixth term it would be 5, etc. Therefore in the rth term the exponent of b would be $r-1$.

The *sign* of any term of the expansion (if n is a positive integer) is plus if the binomial is $a+b$. If the binomial is $a-b$, the terms containing the odd powers of b will be negative; the sign of the rth term being minus if $r-1$ is odd.

Therefore the rth term ($r \neq 1$) of $(a+b)^n$ equals plus *or* minus

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}. \quad (1)$$

The formula for the $(r+1)$st term is more simple and more easily applied. It is plus *or* minus

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{r!}a^{n-r}b^{r}. \quad (2)$$

If we wanted the 12th term, we would in using (1) substitute 12 for r, and in using (2) we would substitute 11 for r.

EXERCISES

Write the:

1. 5th term of $(a+b)^{10}$.

Solution: Substituting 10 for n and 5 for r in the formula (1) gives

$$\frac{10\cdot 9\cdot 8\cdot 7}{4!}a^6b^4 = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}a^6b^4 = 210\,a^6b^4.$$

2. 6th term of $(a+b)^9$.

3. 4th term of $(a+b)^{20}$.

4. 7th term of $(a-b)^{10}$.

5. 8th term of $(a-b)^{15}$.

6. 4th term of $\left(a+\frac{1}{a}\right)^{30}$.

7. 5th term of $(a^2-b)^{20}$.

8. 6th term of $\left(\frac{a}{b}-\frac{b^2}{a}\right)^{18}$.

9. 7th term of $\left(\frac{a^2}{b}-\frac{2\,b^2}{a}\right)^{14}$.

10. middle term of $(x^2-x)^{16}$.

11. 5th term of $\left(\sqrt{x}-\sqrt{\frac{y}{x}}\right)^{15}$.

Find the coefficient of:

12. x^5 in $(1+x)^{10}$.

13. x^8 in $(1+x^2)^{16}$.

14. x^{15} in $(x^3+1)^{15}$.

15. x^{10} in $(x^2-x^{-1})^{14}$.

16. Expand $(3+1)^{-1}$ and $(1+3)^{-1}$ by the binomial theorem, and, if possible, find the sum of each series thus obtained.

17. Treat $(2+1)^{-1}$ and $(1+2)^{-1}$ as in Exercise 16.

CHAPTER XLII

SUPPLEMENTARY TOPICS

217. Theorems on irrational numbers. In order to solve a linear equation or a system of linear equations having rational coefficients, we need use only the operations of addition, subtraction, multiplication, and division. When, however, we attempt to solve the equation of the second degree, $x^2 = 2$, we find that there is no rational number which satisfies it. This last fact can be proved if we assume that

An integral factor of one member of an identity between integers is also a factor of the other member.

This is certainly true. For example, let $2a = b$, where a and b are integers. Then since 2 is a factor of the left member it is also a factor of the right.

THEOREM 1. *No rational number satisfies the equation* $x^2 = 2$.

Proof. Evidently no integer satisfies the equation. Let us make the supposition that a rational fraction in its lowest terms, $\frac{a}{b}$, satisfies it. Then

$$\left(\frac{a}{b}\right)^2 = 2, \tag{1}$$

or

$$a^2 = 2b^2. \tag{2}$$

From (2) it is seen that 2 is a divisor of the left member and therefore a divisor of a^2, and hence a divisor of a. Let us then suppose $a \div 2 = m$, or

$$a = 2m. \tag{3}$$

Then

$$a^2 = 4m^2. \tag{4}$$

From (2) and (4),

$$4m^2 = 2b^2, \tag{5}$$

or

$$2m^2 = b^2. \tag{6}$$

Hence 2 must be a divisor of b^2 and therefore of b. Then 2 is a divisor of both a and b, which contradicts the hypothesis that $\frac{a}{b}$ is a rational fraction in its lowest terms.

Therefore no rational number satisfies the equation $x^2 = 2$.

Note. This theorem, when stated in geometrical language, asserts that the hypotenuse of an isosceles right triangle is not commensurate with the legs of the triangle. In this form the theorem was stated, and perhaps proved, by Pythagoras, about 525 B.C. The proof given here is found in Euclid's "Geometry," and some historians think that it is the very demonstration given by Pythagoras himself, and was inserted by Euclid in his book for its historical interest.

THEOREM 2. *The square root of a rational number cannot be the sum of a rational number and a quadratic surd.*

Proof. Suppose x is a rational number and $\sqrt{a}$ and $\sqrt{b}$ are surds. Then, if possible, suppose

$$\sqrt{a} = \sqrt{b} \pm x. \tag{1}$$

Squaring each member of (1),

$$a = b + x^2 \pm 2x\sqrt{b}. \tag{2}$$

Solving (2), $$\sqrt{b} = \pm\frac{a - b - x^2}{2x}. \tag{3}$$

But (3) is impossible, for it asserts that a surd equals a rational number.

Therefore $\sqrt{a} \neq \sqrt{b} \pm x$ if $\sqrt{a}$ and $\sqrt{b}$ are surds.

THEOREM 3. *If each member of an equation consists of a rational number and a quadratic surd, then the rational parts are equal and the irrational parts are equal.*

Proof. Let $$a + \sqrt{b} = c + \sqrt{d}. \tag{1}$$

If possible, suppose $$c = a \pm x. \tag{2}$$

Then $$a + \sqrt{b} = a \pm x + \sqrt{d}, \tag{3}$$

or $$\sqrt{b} = \pm x + \sqrt{d}. \tag{4}$$

But (4) by the preceding theorem is impossible.

Consequently $a = c$, and hence from (1), $\sqrt{b} = \sqrt{d}$.

Therefore, if $a + \sqrt{b} = c + \sqrt{d}$, $a = c$ and $\sqrt{b} = \sqrt{d}$.

218. Cube root of algebraic expressions. Since by actual multiplication $$(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3,$$

a careful inspection of the expression

$$\boldsymbol{t^3 + 3t^2u + 3tu^2 + u^3}$$

will enable one to extract the cube root of any polynomial which is a perfect cube; for the extraction of cube and other roots is not a mysterious, unreasonable process, but merely an intelligent undoing of the work of multiplication. We see that the first term of the result is the cube root of the first term of the polynomial t^3. The second term of the cube root, u, can be obtained by squaring t, multiplying it by 3, and dividing the result as a trial divisor into the second term of the polynomial, thus obtaining u. Since $t^3 + 3t^2u + 3tu^2 + u^3 = t^3 + (3t^2 + 3tu + u^2)u = (t+u)^3$, we may then form the complete divisor as indicated by the trinomial in parenthesis. A systematic arrangement of the work follows:

Example 1.

$$\begin{array}{ll|l}
 & & t^3 + 3t^2u + 3tu^2 + u^3 \,\underline{|\, t + u} \\
 & & t^3 \\ \hline
\text{Trial divisor,} & 3 \cdot t^2 = 3t^2 & 3t^2u + 3tu^2 + u^3 \\
\text{Second term of root,} & & \\
\quad 3t^2u \div 3t^2 = u & & \\
\text{Complete divisor,} & 3t^2 + 3tu + u^2 & \underline{3t^2u + 3tu^2 + u^3 = (3t^2 + 3tu + u^2)u}
\end{array}$$

Example 2. Extract the cube root of $27a^3 - 8x^3 + 36ax^2 - 54a^2x$.

Solution:

$$\begin{array}{lr|l}
 & & 27a^3 - 54a^2x + 36ax^2 - 8x^3 \,\underline{|\, 3a - 2x} \\
t^3 = (3a)^3 = & & 27a^3 \\ \hline
\text{Trial divisor, } 3t^2 = 3 \cdot (3a)^2 = 27a^2 & & -54a^2x + 36ax^2 - 8x^3 \\
\text{Second term of root, } u \text{, equals} & & \\
\quad -54a^2x \div 27a^2 = -2x & & \\
\quad 3tu = 3 \cdot 3a(-2x) = -18ax & & \\
\quad u^2 = \quad (2x)^2 = 4x^2 & & \\
\text{Complete divisor, } 27a^2 - 18ax + 4x^2 & & \\
 & -2x & \\ \hline
 & -54a^2x + 36ax^2 - 8x^3 & \underline{-54a^2x + 36ax^2 - 8x^3}
\end{array}$$

The student should note particularly the form of the trial divisor and of the complete divisor. They are very important in extracting the cube root of a polynomial or of an arithmetical number.

If t in the preceding example be replaced by the binomial $h + t$, we obtain

$$[(h+t)+u]^3 = (h+t)^3 + 3(h+t)^2u + 3(h+t)u^2 + u^3.$$

If this last were expanded fully, we would obtain a polynomial of ten terms which would be a perfect cube. Its cube root could be obtained as before, first obtaining h and then t.

Then we could regard $h + t$ as a single term, form the trial divisor and the complete divisor as before, and obtain the third term of the root. Thus the method may be extended to any polynomial of more than four terms which is a perfect cube.

RULE. *Arrange the terms of the polynomial according to the powers of some letter in it. Extract the cube root of the first term. Write the result as the first term of the root, and subtract its cube from the given polynomial.*

Square the part of the root already found and multiply the result by 3 for a trial divisor. Divide the first term of this product into the first term of the remainder, and write the quotient as the second term of the root.

Annex to the trial divisor three times the product of the first term and the second term of the root, and the square of the second term also, thus forming the complete divisor.

Multiply the complete divisor by the second term of the root, and subtract the result from the remainder.

If terms of the polynomial still remain, square the part of the root already found, and multiply the result by 3 for a trial divisor. Divide the first term of the trial divisor into the first term of the remainder, write the quotient as the third term of the root, form the complete divisor, and proceed as before until the process ends, or until it has been carried far enough.

EXERCISES

Extract the cube root of:

1. $x^3 + 3x^2 + 3x + 1.$

2. $8x^3 - 12x^2 + 6x - 1.$

3. $27x^3 + 27x^2y + 9xy^2 + y^3.$

4. $64a^3 - 144a^2c + 108ac^2 - 27c^3.$

5. $x^{12} - 15x^{10} + 75x^8 - 125x^6.$

6. $x^6 - \frac{3x^4}{2} + \frac{3x^2}{4} - \frac{1}{8}.$

8. $\frac{a^3}{c^6} - \frac{3}{c^3} - \frac{c^3}{a^6} + \frac{3}{a^3}.$

7. $\frac{x^3}{27} - \frac{2}{3} + \frac{4}{x^3} - \frac{8}{x^6}.$

9. $x^{\frac{3}{2}} + \frac{12x^{\frac{1}{2}}}{a^2} - \frac{6x}{a} - \frac{8}{a^3}.$

10. $a^3 + b^3 - 1 - 3a^2 - 3b^2 - 6ab + 3a^2b + 3ab^2 + 3a + 3b$.

11. $x^6 + 8 - 9x^5 + 66x^2 - 36x + 33x^4 - 63x^3$.

12. Find the sixth root of $x^6 - 12x^5 + 64 - 192x + 240x^2 + 60x^4 - 160x^3$.

13. Find the first three terms in the cube root of $1 + 3x$.

219. Cube root of arithmetical numbers. The process of extracting the cube root of an arithmetical number does not differ greatly from the method of extracting the cube root of any polynomial. The formula for the complete divisor, $3t^2 + 3tu + u^2$, can be used to guide the important steps in the work. The first step, however, is pointing off, the reason for which appears from a study of the following table:

$n =$	1	10	100	1000
$n^3 =$	1	1000	1,000,000	1,000,000,000

From this table it is obvious that the cube root of an integer of three digits or less must contain only *one* digit on the left of the decimal point. Similarly, we see that the cube root of an integer containing four, five, or six digits contains *two* digits on the left of the decimal point; and the cube root of an integer of seven, eight, or nine digits contains *three* digits on the left of the decimal point. Hence in cube root we find it convenient to begin at the decimal point and point off the number in periods of three figures each — to the left if the number is integral, to the right if it is decimal; to both the left and right if the number is part integral and part decimal. There may, of course, be an incomplete period on the left. Zeros should be used to complete any partial period on the right.

If we now imagine $(t + u)^3$ to be a number consisting of a tens and a units digit, we may translate $t^3 + 3t^2u + 3tu^2 + u^3$ thus: *the cube of the tens + 3 times the square of the tens times the units + 3 times the tens times the square of the units + the cube of the units.*

Now subtracting t^3 from the polynomial, we may write the other three terms thus: $(3t^2 + 3tu + u^2)u$. Here the trinomial in parenthesis is the complete divisor. The process of extracting the cube root of 50,653 follows:

$$\begin{array}{llr|l}
 & & & 50'653 \;\underline{|\,30 + 7 = 37} \\
 & t^3 = \quad (30)^3 = & & 27\,000 \\ \cline{4-4}
\text{Trial divisor,} & 3t^2 = 3 \cdot (30)^2 = & 2\,700 & 23\,653 \\
\text{Second term of root, } u, & 23653 \div 2700 = 7 + & & \\
 & 3tu = 3 \cdot 30 \cdot 7 = & 630 & \\
 & u^2 = \quad 7^2 \quad = & \underline{\quad 49} & \\
\text{Complete divisor,} & 3t^2 + 3tu + u^2 = & 3\,379 & \underline{23\,653 = 3\,379 \times 7}
\end{array}$$

To obtain the second term of the root, we divided 23,653 by 2700, which gave almost exactly the number 8. But since the trial divisor 2700 must be increased by $3tu$ and u^2 to form the complete divisor, a moment's thought showed that 8 was too great. This means that the trial divisor is really a *trial divisor*, and its use does not give us with certainty the next term of the root. A little experience will enable one to look ahead and decide mentally on the next root figure. If one decides on a root figure either too great or too small, the product of the complete divisor and this root digit will be too great or too small and the subsequent work will show the error.

Since 374, for example, may be regarded as 37 tens plus 4 units, the process just illustrated may be applied to a number whose cube root contains three digits, as follows:

$$\begin{array}{llr|l}
 & & & 644'972'544 \;\underline{|\,800 + 60 + 4 = 864} \\
t^3 = & (800)^3 = & & 512\,000\,000 \\ \cline{4-4}
3t^2 = & 3\,(800)^2 = & 1\,920\,000 & 132\,972\,544 \\
\multicolumn{3}{l}{\text{Second term of root, } u \text{, equals}} & \\
\multicolumn{3}{l}{132\,972\,534 \div 1\,920\,000 = 60 +} & \\
3tu = & 3\,(800)\,(60) = & 144\,000 & \\
u^2 = & (60)^2 = & 3\,600 & \\ \hline
3t^2 + 3tu + u^2 & = & 2\,067\,600 & \underline{124\,056\,000 = 2\,067\,600 \times 60} \\
3t^2 = & 3\,(860)^2 = & 2{\cdot}218\,800 & 8\,916\,544 \\
\multicolumn{3}{l}{\text{Third term of root,}} & \\
\multicolumn{3}{l}{8\,916\,534 \div 2\,218\,800 = 4 +} & \\
3tu = & 3\,(860)\,4 = & 10\,320 & \\
u^2 = & 4^2 = & 16 & \\ \hline
3t^2 + 3tu + u^2 & = & 2\,229\,136 & \underline{8\,916\,544 = 2\,229\,136 \times 4}
\end{array}$$

EXERCISES

(Obtain roots in Exercises 6–9 correct to three decimals.)

Extract the cube root of:

1. 15625.	**4.** 13481272.	**7.** .0173.
2. 12167.	**5.** 41063.625.	**8.** .004913.
3. 1404928.	**6.** 1.0528.	**9.** .000062.

10. Find $\sqrt[3]{35}$ to three decimal places.

11. Find $\sqrt[3]{\frac{2}{3}}$ to three decimal places.

12. Find the edge of a cube whose volume is 5832 cubic inches.

13. Find the diagonal of a cube whose volume is 46656 cubic meters.

14. Find the sixth root of 46656000.

Note. It may be pointed out that ability to extract cube root is not a real necessity. For in engineering practice, or in any work requiring cube (or higher) roots, it is customary to obtain them from a table of roots, or by means of a slide rule or a table of logarithms. The roots can be obtained in any of these ways far more rapidly than by the method explained in the text.

220. Symmetric functions. An expression involving two letters is said to be **symmetric** in those letters if they may be interchanged without affecting the value of the expression.

For example, $x + y$, $x^2 + y^2$, and xy are symmetric functions of x and y; $x - y$, $x + xy$, and $\dfrac{x}{x+y}$ are not symmetric.

If we let r_1 and r_2 represent the roots of the equation $x^2 + bx + c = 0$, we know that

$$b = -(r_1 + r_2), \quad c = r_1 r_2.$$

Hence the coefficients of a quadratic equation are symmetric functions of the roots. By means of these expressions more complicated symmetrical functions of the roots of a quadratic equation may be found.

EXAMPLES

1. Without solving the equation, find the sum of the squares of the roots of

$$x^2 + 3x - 7 = 0. \qquad (1)$$

Solution: Let the roots of (1) be r_1 and r_2.

$$(r_1 + r_2)^2 = r_1^2 + r_2^2 + 2r_1r_2.$$

But since $r_1 + r_2 = -3$, and $r_1r_2 = -7$, we obtain, by substituting these values,

$$(-3)^2 = r_1^2 + r_2^2 + 2(-7), \text{ or } r_1^2 + r_2^2 = 5.$$

2. Find the equation whose roots are the reciprocals of the roots of the equation $x^2 - 6x - 2 = 0$.

Solution: Let the roots of the given equation be r_1 and r_2.
Then $r_1 + r_2 = 6$ and $r_1r_2 = -2$.
Let the required equation be $x^2 + bx + c = 0$.

Then $$b = -\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \text{ and } c = \frac{1}{r_1r_2}.$$

Now $$-\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = -\frac{r_1 + r_2}{r_1r_2} = -\frac{6}{-2} = 3,$$

and $$\frac{1}{r_1r_2} = \frac{1}{-2} = -\frac{1}{2}.$$

Hence the required equation is

$$x^2 + 3x - \tfrac{1}{2} = 0, \text{ or } 2x^2 + 3x - 1 = 0.$$

EXERCISES

Find without solving the equation:

1. The sum of the squares of the roots of $x^2 + 4x - 7 = 0$.

2. The sum of the cubes of the roots of $x^2 - 3x - 2 = 0$.

3. The sum of the reciprocals of the roots of $x^2 + 7x + 1 = 0$.

4. Find the equation whose roots are the squares of the roots of the equation $x^2 - x - 5 = 0$.

5. Find the equation whose roots are the reciprocals of the roots of the equation $x^2 + x - 3$.

6. If r_1 and r_2 are the roots of the equation $x^2 - 3x + 5$, find the value of $r_1^2r_2 + r_1r_2^2$.

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10,000	4.641

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